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One Dimensional Differential Transform Method for Some Higher Order Boundary Value Problems in Finite Domain

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Abstract

In this work, we focus our study to fifth and sixth order boundary value problems in a finite domain with two point boundary conditions. The analysis is accompanied by testing the method both on linear and nonlinear problems. The numerical results given by this method are compared with the exact solutions and the results by other analytical methods. The results are very accurate as compared to other methods and reveal the complete reliability of the method in finite domains.

Keywords: Boundary value problems, differential transform method, two point boundary conditions

1 Introduction

Physical phenomena are generally modeled as functional equations and for most of these equations, exact solutions are very rear. Alternatively, purely numerical or analytical methods are used to address such issues. The commonly used analytical methods are Adomian decomposition method (ADM) [2], variational iteration (VIM) [3] , homotopy perturbation(HPM) [5], homotopy analysis method (HAM) [6], optimal homotopy asymptotic method (OHAM) [7]etc. Differential Transform Method (DTM) is one of the analytical methods for differential equations. The basic idea was initially introduced by Zhou [8] in 1986. Its main application therein is to solve both linear and nonlinear initial value problems in electrical circuit analysis. This method develops a solution in the form of a polynomial. Though it is based on Taylor series, still it is totally different from the traditional higher order Taylor series method. The DTM is an alternative procedure for getting Taylor series solution of the differential equations. This method reduces the size of computational domain and is easily applicable to many problems. Fifth-order boundary value problems arise in the mathematical modeling of viscoelastic flows [9]. Sixth-order boundary value problems are known to arise in astrophysics, the narrow convicting layers bounded by stable layers which are believed to surround A-type stars that may be modeled by sixth-order boundary value problems [10]. Glatzmaier [11] also studied that dynamo action in some stars may be modeled by such equations. Fifth and sixth order linear and nonlinear problems were solved by Wazwaz [13], while using decomposition method. Noor et al. [19] investigated these type of problems using homotopy perturbation, variational iteration, the iterative method and homotopy perturbation using He's polynomials. S.T. Mohyud-Din et al. [17]used an iterative algorithm for the solution of fifth order boundary value problems. This work includes the solution of fifth and sixth-order boundary value problems by DTM. The method is tested on linear and non-linear problems from literature. The numerical results are compared with the results obtained by decomposition method, homotopy perturbation methods, variational iteration method and the iterative method, to show the effectiveness and accuracy of the method. The structure of this paper is organized as follows. Section 2 is devoted to the basic idea of differential transform method; section 3 is composed of analysis of the proposed method for the problems under discussion. Some numerical examples are presented in Section 4. In Section 5, we concluded by discussing results of the numerical simulation by using Mathematica.5.2.

2 Differential Transform Method(DTM)

If $\phi(x)$ is a given function, its differential transform is defined as

$$\Phi(r) = \left.\frac{d^r\phi(x)}{dx^r}\right|_{x=0} \tag{1}$$

The inverse transform of $\Phi(r)$ is defined by

$$\phi(x) = \sum_{r=0}^{\infty} x^r \Phi(r) \tag{2}$$

For practical application, the function $\phi(x)$ is expressed by a finite series

$$\phi(x) = \sum_{r=0}^{N} x^r \Phi(r) \tag{3}$$

Equation (2) implies that, $\phi(x) = \sum_{r=N+1}^{\infty} x^r \Phi(r)$, is negligibly small. The fundamental operations of the DTM are given in table 1.

Table 1: Fundamental operations

Function	Transformed function
$\phi(x) = k_1 g(x) \pm k_2 h(x)$	$\Phi(r) = k_1 G(r) \pm k_2 H(r)$
$\phi(x) = g^m(x)$	$\Phi(r) = \frac{(m+r)!}{r!} G(m+r)$
$\phi(x) = g(x)h(x)$	$\Phi(r) = \sum_{k=0}^{r} G(k)H(r-k)$
$\phi(x) = x^m$	$\Phi(r) = \delta(r-m), \quad \delta(p) = \begin{cases} 1 & : p = 0 \\ 0 & : p \neq 0 \end{cases}$
$\phi(x) = g_1(x)g_2(x)\cdots g_m(x)$	$\Phi(r) = \sum_{k_{m-1}=0}^{r} \sum_{k_{m-2}=0}^{k_{m-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_m(r-k_{m-1})$

3 Analysis of the method

Consider an nth order boundary value problem of the form

$$y^n(x) = f(x, y, y', y'', \cdots, y^{n-1}), \quad 0 < x < b, \tag{4}$$

with the boundary conditions

$$B(y, \frac{dy}{dx}) = 0. \tag{5}$$

The differential transform of (4) is,

$$Y(r+n) = \frac{F(r)}{\prod_{i=1}^{n}(r+i)}, \tag{6}$$

where $F(r)$ is the differential transform of $f(x, y, y', y'', \cdots, y^{n-1})$.
The transformed boundary conditions of (5) are given by,

$$Y(r) = \alpha,\ Y(m) = \sum_{r=0}^{N} \prod_{i=1}^{m-1} (r-i)Y(r) = \beta_m,\ (m < n), \tag{7}$$

where m is the order of the derivative in the boundary conditions and α, β_m are real constants.
Using equations (6) and (7), values of $Y(i) : i = 1, 2, 3, \cdots$ are obtained and then using the inverse differential transform, the following approximate solution upto $O(x^{N+1})$ is obtained.

$$\widetilde{y} = \sum_{r=0}^{N} x^r Y(r).$$

The error, $E^* = y(x) - \widetilde{y}(x)$ can be approximated by the residual,

$$R = \widetilde{y}^n(x) - f(x, \widetilde{y}, \widetilde{y}', \cdots, \widetilde{y}^{n-1}), \tag{8}$$

whenever the exact solution is not available. We will compare E* and R in the next section. One can insure the validity of the obtained solution by examining R.

4 Numerical Examples

In this section, solutions of different fifth and sixth order boundary value problems are given by using DTM. These problems have been solved by different methods, such as decomposition method, homotopy perturbation method, variational iteration method etc.

Fifth order boundary value problems

Example 1: Consider a linear fifth order boundary value problem

$$y^{(5)}(x) = y(x) - 15e^x - 10xe^x, \quad 0 < x < 1, \tag{9}$$

with the conditions:

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y(1) = 0, y'(1) = -e. \tag{10}$$

Exact solution to this problem is: $y(x) = x(1-x)e^x$.
The differential transform of (9) is:

$$Y(r+5) = \frac{1}{\prod_{i=1}^{5}(r+i)}\left\{Y(r) - \frac{15}{r!} - 10\sum_{k=0}^{r}\frac{\delta(r-k-1)}{k!}\right\} \tag{11}$$

Differential transform of the boundary conditions (10) are:

$$Y(0) = 0, Y(1) = 1, Y(2) = 0, \sum_{r=0}^{N} Y(r) = 0, \sum_{r=0}^{N} rY(r) = -e. \tag{12}$$

Using (11), (12) and the inverse differential transform, the following series solution of the problem, upto $O(x^{15})$, is obtained.

$$\begin{aligned}\widetilde{y}(x) = {} & x - 0.5x^3 - 0.333333335x^4 - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - 0.001190476x^8 \\ & - 0.000173611x^9 - \frac{x^{10}}{45360} - \frac{x^{11}}{403200} - \frac{x^{12}}{3991680} \\ & - 2.29644 \times 10^{-8}x^{13} - 1.92708 \times 10^{-9}x^{14} + O(x^{15}).\end{aligned} \tag{13}$$

Numerical results for solution (13) are given in tables 2 and 3.

Table 2: It shows comparison of the solutions obtained by DTM (13), ADM[12], HPM[14], VIM[16], ITM[17] and VIHPM[15]. From the numerical results it is clear that the differential transform method is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained.

Table 2 for example 1

x	Exact Sol.	DTM Sol.	E*(DTM)	E*(Others)
0	0.00000	0.0000	0.0000	0.0000
0.1	0.099465383	0.099465383	-1.6E-12	-3.0E-11
0.2	0.195424441	0.195424441	-1.1E-11	-2.0E-10
0.3	0.28347035	0.28347035	-3.2E-11	-4.0E-10
0.4	0.358037927	0.358037927	-6.4E-11	-8.0E-10
0.5	0.412180318	0.412180318	-1.0E-10	-1.2E-9
0.6	0.437308512	0.437308512	-1.3E-10	-2.0E-9
0.7	0.422888068	0.422888068	-1.4E-10	-2.2E-9
0.8	0.356086548	0.356086548	-1.2E-10	-1.9E-9
0.9	0.221364280	0.221364280	-5.7E-11	-1.4E-9
1.0	0.0000	0.000	0.000	0.000

E*=Exact solution-analytical solution

Table 3 for example 1.

x	E*	R
0	0.0000	0.0000
0.1	-1.6E-12	7.7E-15
0.2	-1.1E-11	5.6E-12
0.3	-3.2E-11	3.3E-10
0.4	-6.4E-11	5.9E-09
0.5	-1.0E-10	5.5E-08
0.6	-1.3E-10	3.5E-07
0.7	-1.4E-10	1.6E-06
0.8	-1.2E-10	6.3E-06
0.9	-5.7E-11	2.1E-05
1.0	0.000	6.0E-05

Table 3: This table shows Comparison of E* and R where the exact error E* and the residual R are calculated at different mesh points of the problem domain.

Example 2: Consider a nonlinear fifth order boundary value problem

$$y^{(5)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \tag{14}$$

with the conditions:

$$y(0) = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e. \tag{15}$$

Exact solution to this problem is: $y(x) = e^x$.
The differential transform of (14) is:

$$Y(r+5) = \frac{1}{\prod_{i=1}^{5}(r+i)} \sum_{l=0}^{r} \sum_{m=0}^{l} \frac{(-1)^m}{m!} Y(l-m) Y(r-l). \tag{16}$$

Differential transformation of the boundary conditions (15) is:

$$Y(0) = Y(1) = 2Y(2) = 1, \quad \sum_{r=0}^{N} Y(r) = \sum_{r=0}^{N} rY(r) = e. \tag{17}$$

Using (16), (17) and the inverse differential transform, the following series solution of the problem, upto $O(x^{13})$, is obtained.

$$\begin{aligned} \widetilde{y}(x) &= 1 + x + \frac{x^2}{2!} + 0.166667x^3 + 0.0416667x^4 + \frac{x^5}{5!} - \frac{x^6}{6!} \\ &- \frac{x^7}{7!} + 0.0000248016x^8 + 2.75573 \times 10^{-6}x^9 + 2.75573 \times 10^{-7}x^{10} \\ &+ 2.50521 \times 10^{-8}x^{11} + 2.08768 \times 10^{-9}x^{12} + O(x^{13}). \end{aligned} \tag{18}$$

Numerical results for solution (18) are given in table 4.

Table 4: It shows comparison of the DTM solution (18) with the exact solution and the error estimates. Last column of this table exhibits the errors in the solutions by ADM[12], HPM[14], VIM[16], ITM[17] and VIHPM[15] for the same problem. The accuracy of the proposed method can be improved further by adding more terms of the Taylor's series to $\widetilde{y}(x)$.

Table for example 2

x	Exact Sol.	DTM Sol.	E*(DTM)	E*(Others)
0	0.00000	0.0000	0.0000	0.0000
0.1	1.105170918	1.105170918	1.4E-12	1.0E-9
0.2	1.221402758	1.221402758	9.8E-12	2.0E-9
0.3	1.349858808	1.349858808	2.8E-11	1.0E-8
0.4	1.491824698	1.491824698	5.6E-11	2.0E-8
0.5	1.648721271	1.648721271	8.7E-11	3.1E-8
0.6	1.822118800	1.822118800	1.1E-10	3.7E-8
0.7	2.013752707	2.013752707	1.2E-10	4.1E-8
0.8	2.225540928	2.225540928	9.9E-11	3.1E-8
0.9	2.459603111	2.459603111	4.4E-11	1.4E-8
1.0	2.718281828	2.718281828	0.000	0.000

E*=Exact solution-analytical solution

Sixth order boundary value problems

Example 3: Consider a linear sixth order boundary value problem

$$y^{(6)}(x) = y(x) - 6e^x, \quad 0 < x < 1, \tag{19}$$

with the conditions:

$$\begin{aligned} y(0) = -y^{(2)}(0) = -\frac{1}{3}y^4(0) = 1, \\ y(1) = 0,\, y^{(2)}(0) = \frac{1}{2}y^4(0) = -2e. \end{aligned} \tag{20}$$

Exact solution to this problem is: $y(x) = (1-x)e^x$.
The differential transform of (19) is:

$$Y(r+6) = \frac{1}{\prod_{i=1}^{6}(r+i)}\left\{Y(r) - \frac{6}{r!}\right\} \tag{21}$$

Differential transformations of the boundary conditions (20) are:

$$\sum_{r=0}^{N} Y(r) = 0,\, \sum_{r=0}^{N} r(r-1)Y(r) = -2e,\, \sum_{r=0}^{N}\prod_{i=0}^{3} Y(r) = -4e. \tag{22}$$

Table 5: In this table we compare the exact solution with the DTM solution (23) and the errors obtained by decomposition method(ADM)[13], homotopy perturbation method(HPM)[18] and the variational iteration method(VIM)[19].

Table for example 3

x	Exact Sol.	E*(DTM) Sol.	E*(DTM)	E*(ADM)	E*(HPM)	E*(VIM)
0	1	1	0	0	0	0
0.1	0.994653826	0.994653760	6.57E-8	-4.09E-4	-4.09E-4	-4.09E-4
0.2	0.977122206	0.977122081	1.26E-7	-7.78E-4	-7.78E-4	-7.78E-4
0.3	0.944901165	0.944900991	1.74E-7	-1.07E-3	-1.07E-3	-1.07E-3
0.4	0.895094818	0.895094611	2.08E-7	-1.26E-3	-1.26E-3	-1.26E-3
0.5	0.824360635	0.824360414	2.21E-7	-1.32E-3	-1.32E-3	-1.32E-3
0.6	0.728847520	0.728847306	2.14E-7	-1.26E-3	-1.26E-3	-1.26E-3
0.7	0.604125812	0.604125628	1.85E-7	-1.07E-3	-1.07E-3	-1.07E-3
0.8	0.445108186	0.44510805	1.36E-7	-4.09E-4	-4.09E-4	-4.09E-4
0.9	0.245960311	0.24596024	7.23E-8	-7.78E-4	-7.78E-4	-7.78E-4
1.0	0.0000	0	0	0	0	0

E*=Exact solution-analytical solution

Using (21), (22) and the inverse differential transform, the following series solution of the problem, upto $O(x^{13})$, is obtained.

$$\begin{aligned}\widetilde{y}(x) = {} & 1 - 6.66476 \times 10^{-7}x - \frac{x^2}{2} - 0.333332x^3 - \frac{x^4}{8} - 0.0333336x^5 - \frac{x^6}{144} \\ & - 0.00119048x^7 - \frac{x^8}{5760} - 0.0000220458x^9 - \frac{x^{10}}{403200} \\ & - 2.50522 \times 10^{-7}x^{11} - \frac{x^{12}}{43545600} + O(x^{13}). \qquad (23)\end{aligned}$$

Numerical results for solution (23) are given in table 5.

Example 4: Now consider a nonlinear sixth order boundary value problem

$$y^{(6)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \qquad (24)$$

with the conditions:

$$y^{(2k)}(0) = 1, \quad y^{(2k)}(1) = e, \quad k = 0, 1, 2. \qquad (25)$$

Exact solution to this problem is: $y(x) = e^x$.

$$Y(r+6) = \frac{1}{\prod_{i=1}^{6}(r+i)} \sum_{l=0}^{r} \sum_{m=0}^{l} \frac{(-1)^m}{m!} Y(l-m)Y(r-l). \qquad (26)$$

Differential transformations of the boundary conditions (25) are:

$$\begin{aligned} Y(k) &= \frac{1}{k!},\ k = 0, 2, 4. \\ \sum_{r=0}^{N} Y(r) = \sum_{r=0}^{N} r(r-1)Y(r) &= \sum_{r=0}^{N} \prod_{k=0}^{3}(r-k)Y(r) = e. \end{aligned} \qquad (27)$$

Using (26), (27) and the inverse differential transform, the following series solution of the problem, upto $O(x^{13})$, is obtained.

$$\begin{aligned}\widetilde{y}(x) = {} & 1 + x + \frac{x^2}{2!} + 0.166666586x^3 + \frac{x^4}{4!} + 0.008333359x^5 + \frac{x^6}{6!} \\ & + 0.000198413x^7 + 0.000024802x^8 + 2.75573 \times 10^{-6}x^9 + 2.75573 \times 10^{-7}x^{10} \\ & + 2.50523 \times 10^{-8}x^{11} + 2.08768 \times 10^{-9}x^{12} + O(x^{13}). \qquad (28)\end{aligned}$$

Numerical results for solution (28) are given in table 6.

Table 6: It shows comparison of the DTM solution (28) with the exact solution and the errors obtained by decomposition method(ADM)[13], homotopy perturbation method(HPM)[18] and the variational iteration method(VIM)[19]. It is clear from the results that the method we applied is more efficient and accurate than the other methods.

Table for example 4

x	Exact Sol.	E*(DTM) Sol.	E*(DTM)	E*(ADM)	E*(HPM)	E*(VIM)
0	1	1	0	0	0	0
0.1	1.105170918	1.105170918	-5.4E-9	-1.2E-4	-1.2E-4	-1.2E-4
0.2	1.221402758	1.221402758	-1.0E-8	-2.3E-4	-2.3E-4	-2.3E-4
0.3	1.349858808	1.349858808	-1.4E-8	-3.2E-4	-3.2E-4	-3.2E-4
0.4	1.491824698	1.491824698	-1.7E-8	-3.8E-4	-3.8E-4	-3.8E-4
0.5	1.648721271	1.648721271	-1.8E-8	-4.0E-4	-4.0E-4	-4.0E-4
0.6	1.822118800	1.822118800	-1.8E-8	-3.9E-4	-3.9E-4	-3.9E-4
0.7	2.013752707	2.013752707	-1.5E-8	-3.3E-4	-3.3E-4	-3.3E-4
0.8	2.225540928	2.225540928	-1.1E-8	-2.4E-4	-2.4E-4	-2.4E-4
0.9	2.459603111	2.459603111	-6.0E-9	-1.2E-4	-1.2E-4	-1.2E-4
1.0	2.718281828	2.718281828	-1.0E-13	2.0E-9	2.0E-9	2.0E-9

E*=Exact solution-analytical solution

References

[1] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition method, vol. 60 of Fundamental Theories of Physics,* Kluwer Academic Publishers, Boston, mass, USA, 1994.

[2] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, *Computer Math, Appl.,***21(5)**(1991),101-127.

[3] Ji-Huan He, Variational approach to the sixth-order boundary value problems, *Appl. Math. Comput.***143(2-3)**(2003),537-538.

[4] Ji-Huan He, Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering,* **170(3-4)**(1999),257-262.

[5] Ji-Huan He, Homotopy perturbation method for solving boundary value problems,*Phys. Lett.A,***350**(2006),87-88.

[6] Shijun Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman Hall/CRC, Florida, 2004.

[7] V. Marinca, N.Herisanu, I. Nemes, Optimal homotopy asymptotic method with application to thin film flow, *Cent. Eur. J. Phys.,***6(3)**(2008),648-653.

[8] J. K. Zhou, *Deferential Transformation and its Application for Electrical Circuits,* Huazhong University Press, Wuhan China,1986.(in Chinese).

[9] S. Chandrasekhar,*Hydrodynamic and Hydromagnetic Stability,* Clarenden Press Oxfords, 1961: reprinted by Dover Books, New York, (1981).

[10] J. Toomore, J. R. Zahn, J. Latour, E. A. Spiegel, Stellar convection theory II: single-mode study of the second convection zone in A-type stars, *Astrophys. J.,* **207**(1976), 545-563.

[11] G. A. Glatzmaier, Numerical simulations of stellar convection dynamics at the base of the convection zone, Geophys,*Fluid Dynamics,***31**(1985),137-150.

[12] A. M. Wazwaz, The numerical solution of fifth-order boundary value problems by decomposition method, *J. Comp. and Appl. Math.,***136**(2001), 259-270.

[13] A. M. Wazwaz, The numerical solution of sixth-order boundary value problems by the modified decomposition method,*Appl. Math. Comput.,***118**(2001),311-325.

[14] M. A. Noor, S. T. Mohyud-Din. An efficient algorithm for solving fifth-order boundary value problems, *Mathematical and Computer Modeling,***45**(2007), 954-964.

[15] Muhammad Aslam Noor and Syed Tauseef Mohyud-Din, Variational Iteration Method for Fifth-Order Boundary Value Problems Using He's Polynomials, *Mathematical Problems in Engineering,* **doi:10.1155/2008/954794**.

[16] M. A. Noor and S. T.Mohyud-Din, Variational iteration technique for solving higher order boundary value problems, *Applied Mathematics and Computation,***189(2)**(2007),1929-1942.

[17] Syed Tauseef Mohyud-Din, Ahmet Yildirim and M. M. Hosseini, An Iterative Algorithm for Fifth-order Boundary Value Problems, *World Applied Sciences Journal,***8(5)**2010, 531-535.

[18] Muhammad Aslam Noor, Syed Tauseef Mohyud-Din, Homotopy perturbation method for solving sixth-order boundary value problems,*Comput.Math.Appl.,***55**(2008), 2953-2972.

[19] M.A. Noor, S.T. Mohyud-Din, Variational iteration method for solving sixth order boundary value problems, *pre-print* (2007).

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