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The Graph $\triangle_{2n-1}$ is an Induced Subgraph of a Johnson Graph

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Abstract

Induced subgraphs of Johnson graphs (JIS for short) were studied first time in 2010 by R. Naimi and Jaffrey Shaw [3]. They proposed a conjecture " The graph Δ_p is JIS if and only if p is odd ". We have attacked this conjecture and have been able to prove that If p is odd then Δ_p is JIS. Furthermore, we have characterize all graphs of order less than 6 and of size less than 7 into JIS and Non-JIS category.

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1 Introduction

We work with finite, simple and undirected graphs, so whenever we use the term graph it would be understood that the graph under consideration is simple, finite and undirected.

Given positive natural numbers $n \leq N$, the Johnson graph $J(n,\ N)$ is defined as the graph whose vertices are the n-subsets of the set $\{1,\ 2,\ 3, ..., N\}$, where two vertices are adjacent if they share exactly $n-1$ elements. Hence a graph G is isomorphic to an induced subgraph of a Johnson graph if and only if it is possible to assign, for some fixed n, an n-set S_v to each vertex v

of G such that vertices v and w are adjacent if and only if S_v and S_w share exactly $n-1$ elements. When this happens, we say that the family of n-sets $F = \{S_v : v \in V(G)\}$ realizes G as an induced subgraph of a Johnson graph, which we abbreviate by saying G is JIS.

The graph Δ_p consists of a chain of p " consecutively adjacent " triangles, plus one vertex which is connected to the two vertices of degree 2 in the triangle chain.

Theorem 1.1 [**3**]**:** Suppose G is JIS and L and $L^{'}$ are distinct maxcliques in G. Then

(1): L and $L^{'}$ share at most two vertices.

(2): If L and $L^{'}$ share exactly two vertices, then no vertex in $V(L) \setminus V(L^{'})$ is adjacent to a vertex in $V(L^{'}) \setminus V(L)$.

(3): If L and $L^{'}$ share exactly one vertex, then each vertex in either of the two sets $V(L) \setminus V(L^{'})$ and $V(L^{'}) \setminus V(L)$ is adjacent to at most one vertex in the other set.

Theorem 1.2 [**3**]**:** All complete graphs and all cycles are JIS.

Theorem 1.3 [**3**]**:** A graph is JIS if and only if its 2-core is JIS.

Theorem 1.4 [**3**]**:** Suppose $L_1,\ L_2,\ ...,\ L_k$, where k is odd and at least 3, are distinct maxcliques in a graph G such that L_i shares exactly two vertices with L_{i+1} for $1 \leq i \leq k-1$, and L_k shares exactly two vertices with L_1, then G is not JIS.

Theorem 1.5 [**3**]**:** A graph is JIS iff all its connected components are JIS.

Theorem 1.6 [**3**]**:** The complete bipertite graph $K_{2,\ 3}$ and the graph Δ_2 are not JIS.

2 Main Result

Theorem 2.1 : If p is an odd positive integer, then the graph Δ_p is JIS.

Proof : Let $p = 2q - 1$, where $q \in N$
Since $| V(\Delta_p) |= p + 3$
$\Rightarrow | V(\Delta_{2q-1}) |= 2(q + 1)$
Define the sets S_{v_i}, where $i = 1, 2, 3, ..., 2(q + 1)$ as follows:
$| S_{v_i} |= \frac{q+1}{2} + 1$ if q is odd
and $| S_{v_i} |= \frac{q}{2} + 1$ if q is even, for all i
$\cup_{i=1}^{2(q+1)} S_{v_i} = \{1, 2, 3, ..., k, k + 1, ..., q + 3\}$, where $k =| S_{v_i} |$
Take $S_{v_1} = \{ 1, 2, 3, ..., k - 1, k\}$,
$S_{v_2} = \{1, 2, 3, ..., k - 1, k + 1\}$,
$S_{v_{2j-1}} = (S_{v_{2j-2}} \cap S_{v_{2j-3}}) \cup \{r\}$, $j = 2, 3, 4, ..., q$
where r is the element which occurs in $\cup_{i=1}^{2(q+1)} S_{v_i}$ exactly after the last element of $S_{v_{2j-2}}$, if last element of $S_{v_{2j-2}}$ is $q + 3$ then $r = 1$
And $S_{v_{2j}} = (S_{v_{2j-1}} \cup S_{v_{2j-2}}) \setminus \{s\}$, $j = 2, 3, 4, ..., q$
where s is the first common element to both $S_{v_{2j-1}}$ and $S_{v_{2j-2}}$ provided that s is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2j-3}}$
If such s does not exist then we take s as the 2nd common element to both $S_{v_{2j-1}}$ and $S_{v_{2j-2}}$ provided that s is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2j-3}}$
Again, if such s does not exist then we take s as the 3rd common element to both $S_{v_{2j-1}}$ and $S_{v_{2j-2}}$ provided that s is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2j-3}}$
Continue in this way for getting s.
And finally $S_{v_{2(q+1)}} = (S_{v_{2q+1}} \setminus \{t\}) \cup \{k\}$
where t is the first common element to both $S_{v_{2q+1}}$ and $S_{v_{2q}}$ provided that s is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2q-1}}$
If such t does not exist then we take s as the 2nd common element to both $S_{v_{2q+1}}$ and $S_{v_{2q}}$ provided that t is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2q-1}}$
Again, if such t does not exist then we take t as the 3rd common element to both $S_{v_{2q+1}}$ and $S_{v_{2q}}$ provided that t is not the first common element to any two consecutive sets from the sets $S_{v_1}, S_{v_2}, S_{v_3}, ..., S_{v_{2q-1}}$
Continue in this way for getting t.
It is clear from the definition of these k-sets that these sets realize Δ_{2q-1} as

JIS. □

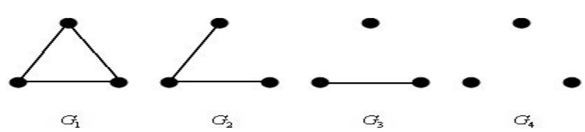
3 JIS-Characterization of All Graphs Whose Order< 6 or Size< 7

We start this section with an obvious result: All null graphs are JIS. Let G be a null graph of order n then
$\{1,\ 2\},\ \{3,\ 4\},\ \{5,\ 6\},\ ...,\ \{2n-1,\ 2n\}$ realize G as JIS.

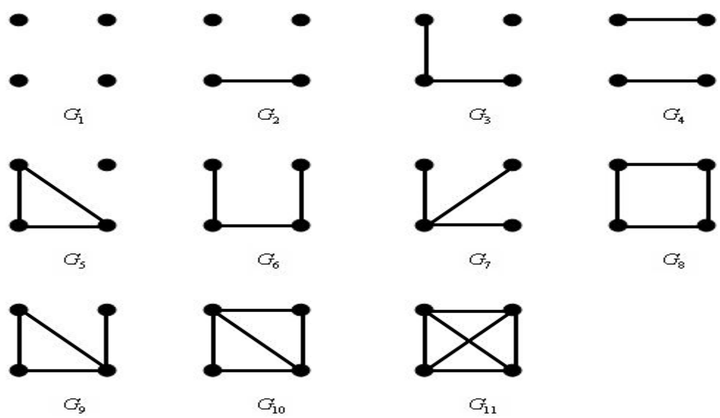
Now we give some results which characterize all graphs of order < 6 and of size < 7 into JIS and non-JIS category.

Theorem 3.1 : All graphs of order less than 5 are JIS.

Proof : Let G be a graph of order $n < 5$
If $n = 1$, then G is complete graph and therefore is JIS.
If $n = 2$, then G is either a null graph or a complete graph
$\Rightarrow G$ is JIS.
If $n = 3$, then G will be isomorphic to one of the following graphs:



G_1 is a complete graph, G_2 is a tree and G_4 is a null graph and so all these three graphs are JIS. Also G_3 is JIS by Theorem 1.3.
If $n = 4$, then G is isomorphic to one of the graphs shown below:

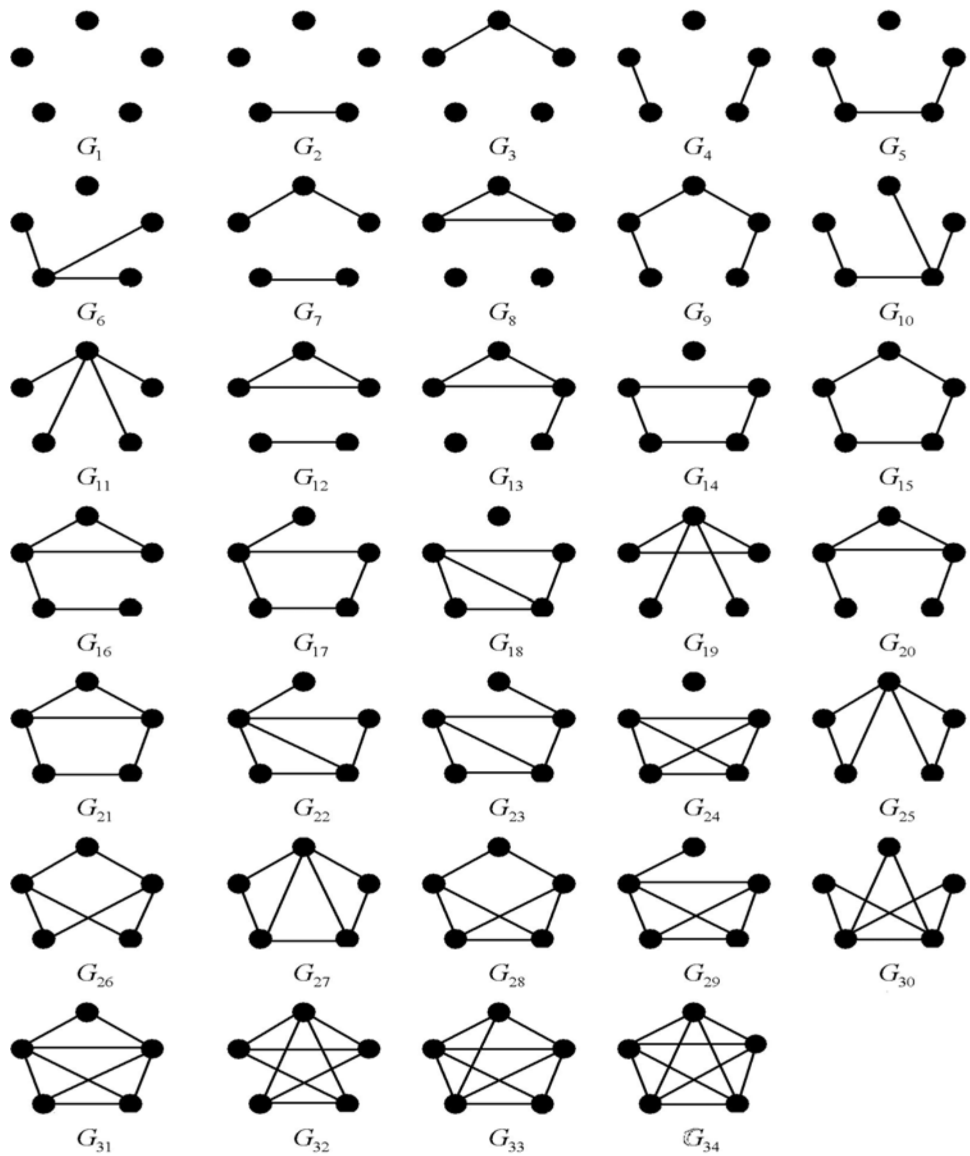


The 2-sets $\{1,\ 2\},\ \{1,\ 3\},\ \{2,3\},\ \{3,\ 4\}$ realize G_{10} as JIS and all the remaining graphs are JIS by Theorems 1.2, 1.3.
Therefore, all graphs of order less than 5 are JIS. □

A Fan graph is denoted by $F_{m,\ n}$ and is defined as the join of two graphs N_m and P_n where N_m is a null graph on m vertices and P_n is a path on n vertices.

Theorem 3.2 : There are only four (non-isomorphic) non-JIS graphs of order 5 namely $K_{2,\ 3}, F_{3,\ 2}, K_5 - e, \triangle_2$

Proof : All non isomorphic graphs of order 5 are shown in the following figure:



Since $G_{26} \cong K_{2,\ 3}$ and $G_{28} \cong \Delta_2$, so by Theorem 1.6 both of these graphs are non-JIS.

Also $G_{30} \cong F_{3,\ 2}$ and $G_{33} \cong K_5 - e$ are not JIS by Theorems 1.4 and 1.1 respectively.

All remaining graphs, except $G_{21},\ G_{25},\ G_{27},\ G_{31},\ G_{32}$, are JIS by Theorems 1.2, 1.3, 1.5.

The 2-sets $\{1,\ 2\},\ \{1,\ 3\},\ \{1,\ 4\},\ \{3,\ 5\},\ \{4,\ 5\}$ realize G_{21} as JIS. The graph G_{25} can be realized by the 2-sets $\{1,\ 2\},\ \{1,\ 3\},\ \{1,\ 4\},\ \{4,\ 5\},\ \{4,\ 6\}$. The graph G_{27} is realized by the 3-sets $\{1,\ 2,\ 5\},\ \{1,\ 3,\ 5\},\ \{1,\ 4,\ 5\},\ \{3,\ 4,\ 5\},\ \{1,\ 3,\ 4\}$ and the 2-sets $\{1,\ 2\},\ \{1,\ 3\},\ \{1,\ 4\},\ \{1,\ 5\},\ \{4,\ 5\}$ realize G_{31} as JIS. Finally, the graph G_{32} can be realized by the $2-$sets $\{1,\ 2\}$, $\{1,\ 3\}$, $\{1,\ 4\}$, $\{2,3\}$, $\{2,4\}$. □

Lemma 3.3 : If G is a connected graph of order n and size m such that $n \geq m$, then G is JIS.

Proof : Since G is connected $\Rightarrow m \geq n-1$. We have two cases ;

Case (i) : If $m = n-1 \Rightarrow G$ is a tree
$\Rightarrow G$ is JIS.

Case (ii) : If $m = n \Rightarrow G$ contains exactly one cycle
If $G = C_n$
By Theorem 1.2 every cycle is JIS.
$\therefore G$ is JIS
If $G \neq C_n$ then after removing vertices of degree 1 from G, we obtain a cycle, say C_k. Therefore using Theorem 1.3 G is JIS. □

Theorem 3.4 : All graphs of size less than 6 are JIS

Proof : Let G be a graph of order n and size $m \leq 5$

Case (i) : If G is connected $\Rightarrow n \leq 6$
If $m \leq n$, then by Lemma 3.3 G is JIS and if $n < m$, then by Theorem 3.1 G is JIS

Case (ii) : If G is disconnected
Suppose that $G_1,\ G_2,\ G_3,\ ...,\ G_k$ be the connected components of G
Let $\mid E(G_i) \mid = m_i$ where $1 \leq i \leq k$, then each $m_i \leq 5$
Therefore, by case (i), each G_i is JIS and hence, by Theorem 1.5, G is JIS. □

Theorem 3.5 : There is only one non-JIS graph of order n and size 6 for each $n \geq 5$.

Proof : Let G be a non-JIS graph of order $n \geq 5$ and size 6.

If $n = 5$, then by Theorem 3.2 $K_{2,\ 3}, F_{3,\ 2}, K_5 - e$, and $\triangle_2$ are the only non-JIS graphs of order 5. But only $K_{2,\ 3}$ has size 6.
Hence the result is true for $n = 5$.
If $n \geq 6$, Since all graphs of size< 6 are JIS (by Theorem 3.4) and $K_{2,\ 3}$ is the only connected non-JIS graph of size 6.
$\Rightarrow K_{2,\ 3}$ must be an induced subgraph of G.
Therefore, G is disconnected having components $K_{2,\ 3}$,$G_6,\ G_7,\ ...,\ G_n$, where each $G_i(6 \leq i \leq n)$ is trivial graph.
i.e., G is obtained by adding $i - 5(6 \leq i \leq n)$ vertices of degree zero to $K_{2,\ 3}$.
But there is, up to isomorphism, only one way to add such $i - 5$ vertices to $K_{2,\ 3}$. Therefore G is unique for each $n \geq 5$. □

REFERENCES

[**1**] C. Vasudev, Graph Theory with Applications, New Age International Publishers, 2006.

[**2**] Geir Agnarsson, Raymond Greenlaw, Graph Theory with Modeling, Applications and Algorithms, Pearson Education inc., 2009.

[**3**] Ramin Naimi, Jeffrey Shaw, Induced Subgraphs of Johnson Graphs, arxiv: 10008.0595[math.CO]v August, 2010.

[**4**] Robin J. Wilson, Introduction to Graph Theory, John Wiley & sons inc., 1985.

[**5**] Terry A. Mckee, F.R. McMorris, Topics in Intersection Graph Theory, SIAM, 1999.

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