Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 43, 2137 - 2144

A Common Fixed Point Theorem for Families of Weakly Compatible Maps on Non-Archimedean Metric Spaces

Mousa Avar

Department of Mathematics, Estahban Branch
Islamic Azad University, Estahban, Iran
m_avar@iauestaban.ac.ir

Abstract

In this paper, we define the concept compatible and weakly compatible for a non-Archimedean metric space and find conditions which under a family of weakly compatible maps on non-Archimedean metric spaces has a unique common fixed point.

Keywords: Common fixed point, compatible, weakly compatible, non-Archimedean metric space

1 Introduction

Let as recall that a non-Archimedean metric a nonempty set X is a non-negative real valued function on $X \times X$ such that for all $x, y, z \in X$.
(i) $x = y$ if and only if $d(x, y) = 0$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

A non-Archimedean metric space is a pair (X, d) such that X is a nonempty set and d is a non-Archimedean metric on X.

Jungck [5] has generalized the notion of commuting maps by introducing the notion of compatible mappings. Moreover, Jungck and Rhoades [6] have introduced the notion of weakly compatible mappings. Here, we define the concepts of compatible and weakly compatible for non-Archimedean metric spaces.

Definition 1.1 *Let A and S be self-maps of a non-Archimedean metric space (X, d).*

(i) *the pair (A, S) is said to be compatible if $d(ASp_n, SAp_n) \to 0$ whenever $\{p_n\}$is a sequence in X such that $d(Ap_n, u) \to 0$and $d(Sp_n, u) \to 0$ for some $u \in X$, as $n \to \infty$.*

(ii) *the pair* (A, S) *is said to be weakly compatible if* $AP = SP$ *for some* $p \in X$*, then* $ASp = SAp$.

Many authors have proved common fixed point theorems for a variety of commuting self-mapping on usual metric, as well as on different kinds of generalized metric space [1, 2, 3, 4, 7]. For example, in [2] Ciric has proved the following common fixed point theorem.

Theorem 1.2 *Let* (X, d) *be a complete, metric space and* $\{T_\alpha\}_{\alpha \in J}$ *be a family of self-mapping on* X*. If there exists a fixed* $\beta \in J$ *such that for each* $\alpha \in J$ *and all* $x, y \in X$

$$d(T_\alpha x, T_\beta y) \leq \lambda \max\{d(x, y), d(x, T_\alpha x), d(y, T_\beta y), 1/2[d(x, T_\beta y) + d(y, T_\alpha x)]\},$$

where $\lambda = \lambda(\alpha) \in (0, 1)$*, then all* T_α *have a unique common fixed point in* X*.*

Singh and Jain [1] have proved the following common fixed point theorem for commuting self -mappings.

Theorem 1.3 *let* A, B, S, T, L *and* M *be self - maps of a complete matric space* (X, d)*, satisfying the conditions .*
(i) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$.
(ii) $AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$.
(iii) *for all* $x, y \in X$ *and for some* $k \in (0, 1)$

$$d(Lx, My) \leq k \max\{d(Lx, ABx), d(My, STy), d(ABx, STy), 1/2[d(Lx, STy) + d(My, ABx)]\}.$$

(iv) *the pair* (L, AB) *is compatible and the pair* (M, ST) *is weakly compatible.*
(v) *either* AB *or* L *is continuous.*
Then A, B, S, T, L *and* M *have a unique common fixed point.*

In this paper, we prove this result for non-Archimedean metric spaces.

2 Main Result

We commence this section with the main result of the paper.

Theorem 2.1 *Let* A, B, S, T, L *and* M *be self-maps on a non–Archimedean complete metric space* (X, d)*. If*
(i) $L(X) \subseteq ST(X)$ *and* $M(X) \subseteq AB(X)$*;*
(ii) $AB = BA$*,* $LB = BL$*,* $ST = TS$ *and* $MT = TM$*;*

(iii) *AB or L is continuous;*
(iv) *the pair (L, AB) is compatible and pair (M, ST) is weakly compatible;*
(v) *there exists $0 < k < 1$ such that for every $u, v \in X$*

$$\begin{aligned} d(Lu, Mv) \leq \; & k \max\{d(ABu, Lu), d(STv, Mv), \\ & d(ABu, STv), d(STv, Lu), d(ABu, Mv))\}, \end{aligned}$$

then A, B, S, T, L, M have a unique common fixed point in X.

Proof. Choose $x_0 \in X$. Then there exists $x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \quad \text{and} \quad Mx_1 = ABx_2 = y_1.$$

Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2k} = STx_{2k+1} = y_{2k} \quad \text{and} \quad Mx_{2k+1} = ABx_{2k+2} = y_{2k+1}$$

for $k \in \mathcal{N}$. From (v) with $u = x_p = x_{2k}$ and $v = x_{q+1} = x_{2m+1}$ we have

$$\begin{aligned} d(y_{2k}, y_{2m+1}) \leq \; & k \max\{d(y_{2k-1}, y_{2k}), d(y_{2m}, y_{2m+1}), d(y_{2k-1}, y_{2m}), \\ & d(y_{2m}, y_{2k}), d(y_{2k-1}, y_{2m+1})\}. \end{aligned}$$

Since X is non–Archimedean, it follows that

$$\begin{aligned} d(y_p, y_{q+1}) \leq \; & k \max\{d(y_{p-1}, y_p), d(y_q, y_{q+1}), d(y_{p-1}, y_q), d(y_q, y_p), d(y_{p-1}, y_{q+1})\} \\ \leq \; & k \max\{d(y_{p-1}, y_p), d(y_q, y_{q+1}), d(y_{p-1}, y_q), d(y_q, y_p), d(y_{p-1}, y_q), \\ & d(y_q, y_{q+1})\}. \end{aligned}$$

If $q = p$, then

$$d(y_p, y_{p+1}) \leq k \max\{d(y_{p-1}, y_p), d(y_p, y_{p+1})\} = kd(y_{p-1}, y_p).$$

So $d(y_{2k}, y_{2k+1}) \leq kd(y_{2k-1}, y_{2k})$. Similarly , $d(y_{2k+1}, y_{2k+2}) \leq kd(y_{2k}, y_{2k+1})$. Hence for each $n \in \mathcal{N}$ we have

$$d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n). \tag{1}$$

It follows that $\{d(y_n, y_{n+1})\}$ is non-increasing. Thus there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = \alpha$. From this and (1) we see that $\alpha = 0$. Thus $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Let $\varepsilon > 0$ be arbitrary. Choose a positive number δ such that $\delta < (\varepsilon - k\varepsilon)/3$ and $kt < k\varepsilon + (\varepsilon - k\varepsilon)/3$, whenever $t \in (\varepsilon, \varepsilon + 2\delta)$. Since $d(y_n, y_{n+1}) \to 0$, there exists an integer $N \geq 1$ such that

$$d(y_{n-2}, y_{n-1}) < \delta \tag{2}$$

for all $n \geq N$. By induction we show that for each $m \geq n \geq N$

$$d(y_n, y_m) < k\varepsilon + \frac{\varepsilon - k\varepsilon}{3} + 2\delta. \tag{3}$$

Fixe $n \geq N$. Obviously, (3) holds for $m = n + 1$. Assuming (3) to hold for an integer $m \geq n + 1$, we shall prove that (3) holds for $m + 1$. We have to consider the following cases.

(I) if $n = 2k$ and $m = 2q$, then $d(y_n, y_m) = d(y_{2k}, y_{2q})$ and $d(y_n, y_{m+1}) = d(y_{2k}, y_{2q+1})$

(II) if $n = 2k$ and $m = 2q+1$, then $d(y_n, y_{m+1}) \leq \max\{d(y_{2k}, y_{2q+1}), d(y_m, y_{m+1})\}$

(III) if $n = 2k+1$ and $m = 2q$, then $d(y_n, y_{m+1}) \leq \max\{d(y_{2k}, y_{2q+1}), d(y_{n-1}, y_n)\}$

(IV) if $n = 2k + 1$ and $m = 2q + 1$, then

$$d(y_n, y_{m+1}) \leq \max\{d(y_{2k}, y_{2q+1}), d(y_{n-1}, y_n), d(y_m, y_{m+1})\} \tag{4}$$

Consider the case (IV). The other cases are similar. Since $d(y_{2k}, y_{2q+1}) = d(Lx_{2k}, Mx_{2q+1})$, by (2), (4) and (v) we have

$$\begin{aligned} d(y_n, y_{m+1}) &\leq k \max\{d(Lx_{2k}, Mx_{2q+1}), \delta, \delta\} \leq k \max\{d(y_{2k-1}, y_{2k}), d(y_{2q}, y_{2q+1}), \\ &\quad d(y_{2k-1}, y_{2q}), d(y_{2q}, y_{2k}), d(y_{2k-1}, y_{2q+1}), \delta\} \\ &\leq kt_{n,m}, \end{aligned} \tag{5}$$

where

$$t_{n,m} = \max\{d(y_{2k-1}, y_{2k}), d(y_{2q}, y_{2q+1}), d(y_{2k-1}, y_{2q}), d(y_{2q}, y_{2k}), d(y_{2k-1}, y_{2q+1})\}.$$

Now we show that

$$d(Lx_{2k}, Mx_{2q+1}) \leq k\varepsilon + (\varepsilon - k\varepsilon))/3, \tag{6}$$

We have

$$d(Lx_{2k}, Mx_{2m+1}) \leq kt_{n,m}. \tag{7}$$

If $n = 2k+1$ and $m = 2q+1$, then by the induction hypotheses $d(y_{2k+1}, y_{2q+1}) < k\varepsilon + (\varepsilon - k\varepsilon)/3 + 2\delta$. It follows from (2) that $d(y_{2k-1}, y_{2k}) = d(y_{n-2}, y_{n-1}) < \delta$ and $d(y_{2q}, y_{2q+1}) = d(y_{m-1}, y_m) < \delta$. From this and (2) we see that

$$\begin{aligned} d(y_{2k-1}, y_{2q}) &\leq \max\{d(y_{2k+1}, y_{2q+1}), d(y_{n-2}, y_{n-1}), d(y_{n-1}, y_n), d(y_{m-1}, y_m)\} \\ &\leq \max\{k\varepsilon + (\varepsilon - k\varepsilon)/3 + 2\delta, \delta\} < \varepsilon + 2\delta. \end{aligned}$$

Hence

$$\begin{aligned} d(y_{2q}, y_{2k}) &\leq \max\{d(y_{2k+1}, y_{2q+1}), d(y_{n-1}, y_n), d(y_{m-1}, y_m)\} \\ &\leq \max\{k\varepsilon + (\varepsilon - k\varepsilon)/3 + 2\delta, \delta, \delta\} < \varepsilon + 2\delta \end{aligned}$$

and

$$\begin{aligned} d(y_{2k-1}, y_{2q+1}) &\leq \max\{d(y_{2k+1}, y_{2q+1}), d(y_{n-2}, y_{n-1}), d(y_{n-1}, y_n)\} \\ &\leq \max\{k\varepsilon + (\varepsilon - k\varepsilon)/3 + 2\delta, \delta, \delta\} < \varepsilon + 2\delta. \end{aligned}$$

Thus $t_{n,m} < \varepsilon+2\delta$ and so $kt_{n,m} < k\varepsilon+(\varepsilon-k\varepsilon)/3$ by (7). Hence $d(Lx_{2p}, Mx_{2q+1}) \leq k\varepsilon + (\varepsilon - k\varepsilon)/3$. Thus we have proved (6). Clearly, from (6) and (7) we infer that

$$d(y_n, y_{m+1}) < k\varepsilon + (\varepsilon - k\varepsilon)/3 + 2\delta.$$

Thus (5) holds. Since $\delta < (\varepsilon-k\varepsilon)/3$, we have $d(y_n, y_m) < \varepsilon$ for all $m \geq n \geq N$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that

$$\begin{aligned} \lim_n y_n &= \lim_k Mx_{2k+1} = \lim_k STx_{2k+1} \\ &= \lim_k Lx_{2k} = \lim_k ABx_{2k} = z. \end{aligned}$$

Now, let AB be continuous. Then $ABx_{2k} \to ABz$ and $ABLx_{2k} \to ABz$. Also, as (L, AB) is compatible, $LABx_{2k} \to ABz$.

(a) From (v) with $u = ABx_{2k}$ and $v = x_{2k+1}$, we have

$$\begin{aligned} d(LABx_{2k}, Mx_{2k+1}) \leq\ & k\max\{d(ABABx_{2k}, LABx_{2k}), d(STx_{2k+1}, Mx_{2k+1}), \\ & d(ABABx_{2k}, STx_{2k+1}), d(STx_{2k+1}, LABx_{2k}), \\ & d(ABABx_{2k}, Mx_{2k+1})\}. \end{aligned}$$

It follows that

$$d(ABz, z) \leq k\max\{d(ABz, ABz), d(z, z), d(ABz, z), d(z, ABz), d(ABz, z)\}.$$

So $d(ABz, z) \leq kd(ABz, z)$. This implies that $d(ABz, z) = 0$. Hence $ABz = z$.

(b) Put $u = z$ and $v = x_{2k+1}$ in condition (v). Then

$$\begin{aligned} d(Lz, Mx_{2k+1}) \leq\ & k\max\{d(ABz, Lz), d(STx_{2k+1}, Mx_{2k+1}), d(ABz, STx_{2k+1}), \\ & d(STx_{2k+1}, Lz), d(ABz, Mx_{2k+1})\}. \end{aligned}$$

Thus

$$\begin{aligned} d(Lz, z) &\leq k\max\{d(z, Lz), d(z, z), d(z, z), d(z, Lz), d(z, z)\} \\ &= kd(Lz, z). \end{aligned}$$

This implies that $d(Lz, z) = 0$. Therefore, $Lz = ABz = z$.

(c) From (v) with $u = Bz$ and $v = x_{2k+1}$, condition (ii), we see that

$$\begin{aligned} d(LBz, Mx_{2k+1}) \leq\ & k\max\{d(ABBz, LBz), d(ABBz, TSx_{2k+1}), d(TSx_{2k+1}, \\ & Mx_{2k+1}), d(TSx_{2k+1}, LBz), d(ABBz, Mx_{2k+1})\}. \end{aligned}$$

Hence

$$\begin{aligned} d(Bz,z) &\leq k\max\{d(Bz,Bz),d(z,z),d(Bz,z),d(z,Bz),d(Bz,z)\} \\ &= kd(Bz,z). \end{aligned}$$

This shows that $Bz = z$. Continuing this procedure, we obtain $Lz = Az = Bz = z$. By condition (i), there exists $v \in X$ such that $z = Lz = STv$.

(d) Putting $u = x_{2k}$ in condition (v), we have

$$\begin{aligned} d(Lx_{2k},Mv) &\leq k\max\{d(ABx_{2k},Lx_{2k}),d(STv,Mv), \\ &\quad d(ABx_{2k},STv),d(STv,Lx_{2k}),d(ABx_{2k},Mv)\}. \end{aligned}$$

So

$$\begin{aligned} d(z,Mv) &\leq k\max\{d(z,z),d(z,Mv),d(z,z),d(z,z),d(z,Mv)\} \\ &= kd(z,Mv). \end{aligned}$$

Hence $Mv = z$ and therefore $STv = Mv = z$. As (M,ST) is weakly compatible, we have $STMv = MSTv$. Thus $STz = Mz$.

(e) Putting $u = x_{2k}$ and $v = z$ in condition (v), we have

$$\begin{aligned} d(Lx_{2k},Mz) &\leq k\max\{d(ABx_{2k},Lx_{2k}),d(STz,Mz), \\ &\quad d(ABx_{2k},STz),d(STz,Lx_{2k}),d(ABx_{2k},Mz)\}. \end{aligned}$$

Thus

$$\begin{aligned} d(z,Mz) &\leq k\max\{d(z,z),d(Mz,Mz),d(z,Mz),d(Mz,z)),d(z,Mz)\} \\ &= kd(z,Mz). \end{aligned}$$

So, $STz = Mz = z$.

(f) Putting $u = x_{2k}$ and $v = Tz$ in condition (v), we have

$$\begin{aligned} d(Lx_{2k},MTz) &\leq k\max\{d(ABx_{2k},Lx_{2k}),d(STTz,MTz), \\ &\quad d(ABx_{2k},STTz),d(STTz,Lx_{2k}),d(ABx_{2k},BTz)\}. \end{aligned}$$

Then

$$\begin{aligned} d(z,Tz) &\leq k\max\{d(z,z),d(Tz,Tz),d(z,Tz)),d(Tz,z),d(Tz,z)\} \\ &= kd(z,Tz). \end{aligned}$$

Therefore $Tz = z$. Continuing this procedure, we have $Mz = Sz = Tz$. Thus we have proved

$$Lz = Mz = Az = Bz = Sz = Tz = z.$$

If L is continuous, then $L^2x_{2k} \to Lz$. Since (L, AB) is compatible, we have $ABLx_{2k} \to Lz$.

(g) Putting $u = Lx_{2k}$ and $v = x_{2k+1}$ in condition (v), we have

$$\begin{aligned} d(L^2x_{2k}, Mx_{2k+1}) \leq\ & k \max\{d(ABLx_{2k}, L^2x_{2k}), d(STx_{2k+1}, Mx_{2k+1}), \\ & d(ABLx_{2k}, STx_{2k+1}), d(STx_{2k+1}, L^2x_{2k})), \\ & d(ABLx_{2k}, Mx_{2k+1}))\}. \end{aligned}$$

Hence

$$\begin{aligned} d(Lz, z) &\leq k \max\{d(Lz, Lz), d(z, z), d(Lz, z), d(z, Lz), d(Lz, z)\} \\ &= kd(Lz, z). \end{aligned}$$

Therefore $Lz = z$. Now , using step (d) , (e) and (f) and continuing step (f) gives us $Mz = Sz = Tz = z$.

(h) By condition (i), there exists $w \in X$ such that $z = Mz = ABw$. Putting $u = w$ and $v = x_{2k+1}$ in condition (v), we have

$$\begin{aligned} d(Lw, Mx_{2k+1}) \leq\ & k \max\{d(ABw, Lw), d(STx_{2k+1}, Mx_{2k+1}), d(ABw, STx_{2k+1}), \\ & d(STx_{2k+1}, Lw), d(ABw, Mx_{2k+1})\}. \end{aligned}$$

Thus

$$d(Lw, z) \leq k \max\{d(z, Lw), d(z, z), d(z, z), d(z, Lw), d(z, z)\} = kd(z, Lw).$$

This implies that $Lw = z = ABw$. As (L, AB) is weakly compatible, we have $Lz = ABz = z$. Similarly to in step (c) it can be shown that $Az = Bz = Lz = z$. Thus we have proved that

$$Lz = Mz = Az = Bz = Sz = Tz = z.$$

Let

$$L\acute{z} = M\acute{z} = A\acute{z} = B\acute{z} = S\acute{z} = T\acute{z} = \acute{z}$$

for some $\acute{z} \in X$. Putting $u = z$ and $v = \acute{z}$ in condition (v), we have

$$\begin{aligned} d(Lz, M\acute{z}) \leq\ & k \max\{d(ABz, Lz), d(ST\acute{z}, M\acute{z}), d(ABz, AB\acute{z}), d(ST\acute{z}, Lz), \\ & d(ABz, M\acute{z})\} = kd(z, \acute{z}). \end{aligned}$$

This means that $d(z, \acute{z}) \leq kd(z, \acute{z})$, thus $z = \acute{z}$ and this show that z is a unique common fixed point of the maps.

References

[1] G. V. R. Babu and K. N. V. V. Vara Prasad, Common fixed point theorems of different compatible type mappings using cirics contraction type condition, *Math. Commun.,* **11** (2006), 87-102.

[2] R. Chen and H. M. He, Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space, *Appl. Math. Lett.*, **20** (2007), 751-757.

[3] R. Chen. and Y. Song, Convergence to common fixed point of nonexpansive semigroups, *Journal of Computational and Applied Mathematics,* **200** (2007), 566-575.

[4] Lj. B. Ciric, Generalized contractions and fixed-point theorems,*Publ. Inst. Math.* **26** (1971), 19-26.

[5] Lj. B. Ciric, On a family of contractive maps and fixed points, *Publ. Inst. Math.* **31** (1974), 45-51.

[6] Lj. B. Ciric and J. S. Ume, Some common fixed point theorems for weakly compatible mappings, *J. Math. Anal. Appl.* **314** (2006), 488-499.

[7] A. Razani and M. Shirdaryazdi, A common fixed point theorem of compatible maps in Menger space, *Chaos, Solitons Fractals.* **32** (2007), 26-34.

Received: July, 2012