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# On the Stability of an Iteration in Cone Metric Spaces

**Bahmann Yousefi and Gholam Reza Moghimi**

Department of Mathematics, Payame Noor University
P.O. Box: 19395-3697, Tehran, Iran
b_yousefi@pnu.ac.ir, Moghimimath@gmail.com

**Abstract**

In this paper we give necessary conditions for the semistability of an iteration procedure in cone metric spaces.

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## 1 Introduction

Let $E$ be a real Banach space. A subset $P \subset E$ is called a cone in $E$ if it satisfies the following:
(i) $P$ is closed, nonempty and $P \neq \{0\}$.
(ii) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space $E$ can be partially ordered by the cone $P \subset E$, by defining; $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in int\ P$, where $int\ P$ denotes the interior of $P$. A cone $P$ is called normal if there exists a constant $k > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$.

In the following we suppose that $E$ is a real Banach space, $P$ is a cone in $E$ and $\leq$ is a partial ordering with respect to $P$.

**Definition 1.1** *([1]) Let $X$ be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies the following:*
*(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,*
*(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,*
*(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*
*Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.*

If $T$ is a self-map of $X$, then by $F(T)$ we mean the set of fixed points of $T$. Also, $\mathbf{N_0}$ will denotes the set of nonnegative integers, i.e., $\mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$.

**Lemma 1.2** *([3]) Let $P$ be a normal cone, and let $\{a_n\}$ and $\{b_n\}$ be sequences in $E$ satisfying the following inequality:*

$$a_{n+1} \leq h a_n + b_n,$$

*where $h \in (0, 1)$ and $b_n \to 0$ as $n \to \infty$. Then $\lim_n a_n = 0$.*

**Definition 1.3** *A self-map $T$ of $(X, d)$ is called nonexpansive if*

$$d(Tx, Ty) \leq d(x, y)$$

*for all $x, y \in X$.*

**Definition 1.4** *A self-map $T$ of $(X, d)$ is called affine if*

$$T(\alpha x + (1 - \alpha)y) = \alpha Tx + (1 - \alpha)Ty$$

*for all $x, y \in X$, and $\alpha \in [0, 1]$.*

For some sources on this topics see [1–7].

# 2 Main Result

Let $(X, d)$ be a cone metric space and $T$ be a self-map of $X$. Let $x_0$ be a point of $X$, and assume then $x_{n+1} = f(T, x_n)$ is an iteration procedure involving $T$, which yields a sequence $\{x_n\}$ of points from $X$. The stability of Picard's iteration, $x_{n+1} = Tx_n$, has been studied in metric spaces in [5]. Here we want to investigate the stability of the iteration

$$x_{n+1} = f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n T z_n$$

where

$$z_n = (1 - \beta_n)x_n + \beta_n T x_n$$

and $\alpha_n, \beta_n \in [0, 1]$.

**Definition 2.1** *Let $X$ be a vector space over the field $F$. Assume that the function $p : X \to E$ having the properties:*

*(a) $p(x, y) \geq 0$ for all $x, y$ in $X$.*

*(b) $p(x + y) \leq p(x) + p(y)$ for all $x, y$ in $X$.*

*(c) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in F$ and $x \in X$.*

*Then $p$ is called a cone seminorm on $X$. A cone norm is a cone seminorm $p$ such that*

*(d) $x = 0$ if $p(x) = 0$.*

We will denote a cone norm by $\|.\|_c$ and $(X, \|.\|_c)$ is called a cone normed space. Also, $d_c(x, y) = \|x - y\|_c$ defines a cone metric on $X$.

**Definition 2.2** *Let $T$ be a self-map of a metric space $(X, d)$. An iteration procedure $x_{n+1} = f(T, x_n)$ is said $T$-semistable if $\{x_n\}$ converges to a fixed point $q$ of $T$, and whenever $\{y_n\}$ is a sequence in $X$ with*

$$\lim_n d(y_{n+1}, f(T, y_n)) = 0$$

*and*

$$d(y_n, f(T, y_n)) = o(t_n)$$

*for some sequence $\{t_n\} \subset R^+$, we have $y_n \to q$.*

**Theorem 2.3** *Let $(X, \|.\|_c)$ be a cone normed space with respect to a normal cone $P$ in the real Banach space $E$, and $T$ be an affine nonexpansive self-map of $X$. Consider the iteration procedure*

$$x_{n+1} = f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n T z_n$$

*where*

$$z_n = (1 - \beta_n)x_n + \beta_n T x_n$$

*and $\alpha_n, \beta_n \in [0, 1]$. If there exist $a \geq 0$ and $b \in (0, 1)$ such that*

$$d_c(f(T, y_n), q) \leq a\ d_c(f(T, y_n), y_n) + b\ d_c(y_n, q) \qquad (*)$$

*for all sequences $\{y_n\}$ with*

$$d_c(Ty_n, y_n) = o\left(\frac{1}{\alpha_n(1 + \beta_n)}\right),$$

*and all $q \in F(T)$, then the given iteration is $T$-semistable.*

**Proof.** First, note that $(*)$ implies that $F(T)$ is a singleton. Indeed, if $p$ and $q$ belong to $F(T)$, then by setting $y_n = p$ in $(*)$ for all $n$, we get $d_c(p, q) \leq b\ d_c(p, q)$. This implies that $p = q$. Now let $F(T) = \{q_0\}$ and $\{y_n\} \subseteq X$ be such that

$$\lim_n d_c(y_{n+1}, f(T, y_n)) = \lim_n \alpha_n(1 + \beta_n)\ d_c(Ty_n, y_n) = 0.$$

Now we show that $y_n \to q_0$. To see this note that by using the notation $(*)$ we have:

$$\begin{aligned} d_c(y_{n+1}, q_0) &\leq d_c(y_{n+1}, f(T, y_n)) + d_c(f(T, y_n), q_0) \\ &\leq d_c(y_{n+1}, f(T, y_n)) + a\ d_c(f(T, y_n), y_n) + b\ d_c(y_n, q_0) \\ &= c_n + b\ d_c(y_n, q_0), \end{aligned}$$

where

$$c_n = d_c(y_{n+1}, f(T, y_n)) + a\ d_c(f(T, y_n), y_n).$$

By Lemma 1.2, it is sufficient to show that

$$c_n \to 0.$$

For this we show that $d_c(f(T, y_n), y_n) \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}
d_c(f(T, y_n), y_n) &= \|f(T, y_n) - y_n\|_c \\
&= \|\ (1-\alpha_n)y_n + \alpha_n T(z_n) - y_n\ \|_c \\
&= \alpha_n\ \|\ Tz_n - y_n\ \|_c \\
&= \alpha_n\ \|\ T((1-\beta_n)y_n + \beta_n Ty_n) - y_n\ \|_c \\
&\leq \alpha_n[(1-\beta_n)\ \|\ Ty_n - y_n\ \|_c + \beta_n\ \|\ T^2y_n - y_n\ \|_c] \\
&= \alpha_n(1-\beta_n)\ d_c(Ty_n, y_n) + \alpha_n\beta_n\ d_c(T^2y_n, y_n) \\
&\leq \alpha_n(1-\beta_n)\ d_c(Ty_n, y_n) + \alpha_n\beta_n\ d_c(T^2y_n, y_n) \\
&\quad + \alpha_n\beta_n\ d_c(Ty_n, y_n) \\
&\leq \alpha_n(1-\beta_n)\ d_c(Ty_n, y_n) + 2\alpha_n\beta_n\ d_c(Ty_n, y_n) \\
&= [\alpha_n(1-\beta_n) + 2\alpha_n\beta_n]\ d_c(Ty_n, y_n) \\
&= \alpha_n(1+\beta_n)\ d_c(Ty_n, y_n)
\end{aligned}$$

which tends to 0 since

$$d_c(Ty_n, y_n) = o\left(\frac{1}{\alpha_n(1+\beta_n)}\right).$$

Thus $y_n \to q_0$ and so the iteration $x_{n+1} = f(T, x_n)$ is T-semistable. So the proof is complete. □

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