Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 26, 1259 - 1268

Property(ω) and Hypercyclic/ Supercyclic Operators

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Abstract

Let T be a bounded linear operator acting on a complex separable infinite-dimensional Banach space X. Denote by $T^\star$ the adjoint of T, We say that T satisfies the property (ω) if : $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$ where $\sigma_a(T)$, $\sigma_{uw}(T)$ $\pi_{00}(T)$ are respectively the approximate point spectrum, the upper semi-Weyl spectrum and the set of isolated eigenvalues of finite multiplicities of T. In the present paper we show that if T is supercyclic/hypercyclic then T satisfies the property(ω) if and only if $\pi_{00}(T) = \pi_{00}(T^\star)$. Also if X is a separable Hilbert space and T satisfies the property(ω), we give the necessary and sufficient conditions for T to be in the norm-closure of the class of hypercyclic(supercyclic)operators.

Keywords: Weyl's theorem, Browder's theorem, property(ω) hypercyclic/supercyclic operators

1 Introduction, Notation and Terminology

Throughout this paper $\mathcal{B}(X)$ denots the Banach algebra of all bounded operators acting in an infinite-dimensional complex Banach space X. For an operator $T \in \mathcal{B}(X)$, T is said to be cyclic if there is a vector $x_0 \in X$, called cyclic vector, such that the orbit $Orb(T, x_0) = (T^k x_0)_{k\geq 0}$ has dense linear span $\overline{Vect(Orb(T, x_0))} = X$. T is said supercyclic if there exists a vector $x_0 \in X$ called supercyclic vector for T such that the set of scalar multiples of the orbit is dense, in the case that $\overline{Orb(T, x_0)} = X$. T is said hypercyclic (x_0 called hypercyclic vector for T), clearly if T is hypercyclic or supercyclic then X is separable. We denote by $\mathcal{HP}(X)$ (resp $\mathcal{SP}(X)$) the set of all hypercyclic (resp supercyclic) operator in $\mathcal{B}(X)$ and $\overline{\mathcal{HP}(X)}$ (rep $\overline{\mathcal{SP}(X)}$) the norm-closure of

the class $\mathcal{H}P(X)$ (resp $\mathcal{S}P(X)$), evidently $\mathcal{H}P(X) \subset \mathcal{S}P(X)$. The first example of a hypercyclic operator was constructed by Rolewicz [18], he showed hat if T is the backword shift on $l^2(IN)$ then λT is hypercyclic if and only if $|\lambda| > 1$, the first sufficient condition for hypercyclicity (hypercyclic criterion) discovered independently by Kitai [13] and Gethner and Shopiro [9]. Recently Salas [19] gave a characterization of supercyclic bilateral backword weighted shifts via the supersyclicity criterion (ie a sufficient condition for supercyclicity). Feldman, V. Miller and L.Miller [8] gave new supercyclicity criteria.
For $T \in \mathcal{B}(X)$ let $N(T)$, $T(X)$, $\sigma(T)$ and $T^\star$ denote respectively the kernel, the range, the spectrum, and adjoint of T. Let $\alpha(T) = dimN(T)$ and $\beta(T) = codimT(X)$ be the nullity and the deficiency of T respectively.
$T \in \mathcal{B}(X)$ is called a semi-Fredholm operator if $T(X)$ is closed and $\alpha(T) < \infty$ or $\beta(T) < \infty$, in the sequel $\rho_{sF}(T)$ will denote the semi-Fredholm resolvent set and the index of T is defined by: $ind(T) = \alpha(T) - \beta(T)$, if both $\alpha(T)$ and $\beta(T)$ are finite then T is a Fredholm operator.
An operator T is called Weyl operator if it is Fredholm operator of index zero, the descent $q(T)$ and the ascent $p(T)$ are given by $q(T) = inf\{n : R(T^n) = R(T^{n+1}\}$ and $p(T) = inf\{n : N(T^n) = N(T^{n+1}\}$ where the infimum over the empty set is taken ∞, it is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ [1], [14].
$T \in \mathcal{B}(X)$ is called Browder operator if it is Fredholm operator of finite ascent and descent, the Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by [1]:
$\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not Weyl operator $\}$ and
$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not Browder operator $\}$
Evidently $\sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$. We denote by $\mathcal{H}(\sigma(T))$ (resp $\mathcal{H}_c(\sigma(T))$)the set of all complex-valued functions which are analytic (resp analytic and non constant) in a neighborhood of the spectrum $\sigma(T)$, for $f \in \mathcal{H}(\sigma(T))$ the operator $f(T)$ is defined by the classical functional calculus. Γ (resp $\Gamma_r, r \geq 0$) denote the unit circle (resp circle of radius r). In the case that X is Hilbert space the following simple spectral description of the $\overline{\mathcal{H}P(X)}$ and $\overline{\mathcal{S}P(X)}$ is due to Herrero [11].

Theorem 1.1. *[11, Theorem(2-1)].*
$\overline{\mathcal{H}P(X)}$ *is the class of all those operators* $T \in \mathcal{B}(X)$ *satisfying the conditions:*
1) $\sigma_w(T) \bigcup \Gamma$ *is connected*
2) $\sigma(T) \backslash \sigma_b(T) = \emptyset$
3) $ind(\lambda I - T) \geq 0$ *for every* $\lambda \in \rho_{sF}(T)$

Theorem 1.2. *[11, Theorem(3-3)].*
$\overline{\mathcal{S}P(X)}$ *is the class of all those operators* $T \in \mathcal{B}(X)$ *satisfying the conditions:*

1) $\sigma(T)\bigcup\Gamma_r$ *is connected (for same* $r \geq 0$ *)*
2) $\sigma_w(T)\bigcup\Gamma_r$ *is connected (for same* $r \geq 0$*)*
2) either $\sigma(T)\backslash\sigma_b(T) = \emptyset$ *or* $\sigma(T)\backslash\sigma_b(T) = \{\alpha\}$ *for same* $\alpha \neq 0$
3) $ind(\lambda I - T) \geq 0$ *for every* $\lambda \in \rho_{sF}(T)$

In [15] V.Miller and L.Miller proved the following corollary.

Corollary 1 [15, Corollary1]
Suppose that X is separable and $T \in \mathcal{B}(X)$, if $\lambda I - T$ is surjective and $\bigcup_{k\geq 0} N(\lambda I - T)^k$ is dense for some λ, then $\varphi(T)$ is supercyclic whenever $\varphi \in \mathcal{H}_c(\sigma(T))$, if G is the component of $\rho_{su}(T)$ containing λ, and if $\varphi(G)\bigcap\Gamma \neq \emptyset$ then $\varphi(T)$ is hypercyclic, $\rho_{su}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is surjectif }\}$.

Denote the following classes of operators , see [1], [2], [3],
$\Phi_+(X) := \{T \in \mathcal{B}(X) : \alpha(T) < \infty$ and $T(X)$ is closed $\}$ the class of all upper semi-Fredholm operators,
$\Phi_-(X) := \{T \in \mathcal{B}(X) : \beta(T) < \infty\}$ the class of all lower semi-Fredholm operators,
$\Phi_\pm(X) := \Phi_+(X)\bigcup\Phi_-(X)$ the class of all semi-Fredholm operators,
$\Phi(X) := \Phi_+(X)\bigcap\Phi_-(X)$ the class of all Fredholm operators,
$\mathcal{B}_+(X) := \{T \in \mathcal{B}(X) : p(T) < \infty\}$ the class of all upper semi-Browder operators,
$\mathcal{B}_-(X) := \{T \in \mathcal{B}(X) : q(T) < \infty\}$ the class of all lower semi-Browder operators,
$\mathcal{B}_0(X) := \mathcal{B}_+(X)\bigcap\mathcal{B}_-(X)$ the class of all Browder operators,
$\mathcal{W}_+(X) := \{T \in \Phi_+(X) : ind(T) \leq 0\}$ the class of all upper semi-Weyl operators,
$\mathcal{W}_+(X) := \{T \in \Phi_-(X) : ind(T) \geq 0\}$ the class of all lower semi-Weyl operators,
$\mathcal{W}(X) := \mathcal{W}_+(X)\bigcap\mathcal{W}_-(X)$ the class of all Weyl operators ,
These classes of operators motivate the following spectra :
$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\}$ the upper semi-Browder spectrum of T
$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\}$ the lower semi-Browder spectrum of T
$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_0(X)\}$ the Browder spectrum of T
$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_+(X)\}$ the upper semi-Weyl spectrum of T
$\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_-(X)\}$ the lower semi-Weyl spectrum of T
$\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}(X)\}$ the Weyl spectrum of T
we have: $\sigma_w(T) = \sigma_w(T^\star)$; $\sigma_b(T) = \sigma_b(T^\star)$ and $\sigma_{ub}(T) = \sigma_{lb}(T^\star)$, $\sigma_{lb}(T) = \sigma_{ub}(T^\star)$, $\sigma_{uw}(T) \subseteq \sigma_{ub}(T)$, and $\sigma_{lw}(T) \subseteq \sigma_{lb}(T)$.

Recall that $T \in \mathcal{B}(X)$ is said to be bounded below if T is injective and

has closed range, the approximate point spectrum of T denoted by $\sigma_a(T)$, $\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below }\}$, we have, see [1] [14]:
$\sigma_{uw}(T) \subseteq \sigma_a(T)$ and $\sigma_{lw}(T) \subseteq \sigma_a(T^\star)$.
We say that $T \in \mathcal{B}(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f = 0$, we say that T has the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$, the SVEP was introduced by Dunford, it plays an important role in local spectral theory [1] [14].
Evidently T has the SVEP at every point of the resolvent $\rho(T) = \mathbb{C}\backslash\sigma(T)$ and both T, $T^\star$ have the SVEP at the points of the topological boundary $\partial\sigma(T)$ of the spectrum, in particular at every isolated point of $\sigma(T)$, ($\lambda_0 \in$ iso $\sigma(T)$).
We have the following implications [1]:
$\lambda_0 \in$ iso $\sigma_a(T) \Longrightarrow T$ has SVEP at λ_0 (1)
In particular if the point spectrum $\sigma_p(T)$ is empty then T satisfied the SVEP.
$p(\lambda_0 I - T) < \infty \Longrightarrow T$ has SVEP at λ_0 (2)
Dually, $q(\lambda_0 I - T) < \infty$ implies $T^\star$ has SVEP at λ_0 (3)
The implications (1), (2) and (3) are equivalences if we assume that
$\lambda_0 I - T) \in \Phi_\pm(X)$ [2].
For $T \in \mathcal{B}(X)$, let us consider the set of all Riesz points :
$p_{00}(T) = \sigma(T)\backslash\sigma_b(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder operator }\}$
let as denote:
$p^a_{00}(T) = \sigma_a(T)\backslash\sigma_{ub}(T) = \{\lambda \in \sigma_a(T) : \lambda I - T \in \mathcal{B}_+(X)\}$.
$\pi_{00}(T) = \{\lambda \in \text{ iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$ the set of isolated eigenvalues of finite multiplicities, and $\pi^a_{00}(T) = \{\lambda \in \text{ iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\}$ It should be noted that:
$p_{00}(T) \subseteq p^a_{00}(T) \subseteq \pi^a_{00}(T)$, $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T)$ and $p_{00}(T) = p_{00}(T^\star)$.
Harte and W.Y. Lee [10] was introduced the Browder's theorem
and a-Browder's theorem : T is said to be satisfying Browder's theorem if
$\sigma_w(T) = \sigma_b(T)$ or equivalently
$\sigma(T)\backslash\sigma_w(T) = p_{00}(T)$, and T is said to be satisfy a-Browder's theorem if $\sigma_{uw}(T) = \sigma_{ub}(T)$ or equivalently $\sigma_a(T)\backslash\sigma_{uw}(T) = p^a_{00}(T)$.
In [1] we have a-Browder's theorem $\Longrightarrow$ Browder's theorem
In [2] it is given that: T or $T^\star$ has SVEP $\Longrightarrow$ a-Browder's theorem hold for both T, $T^\star$
Coburn [6] introduced the Weyl's theorem : T is said to satisfy Weyl's theorem if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$, this equality is the properly proved by Weyl in the case where T is a hermitian operator acting on Hilbert space [20], and after it has been extend to several classes of Hilbert spaces and Banach spaces operators [16].

Rakocevic [17] was introduced two variants of Weyl's theorem, the so called a-Weyl's theorem and the property (ω) studied also by Aiena, Pena and Gillen in [2], [3], [4]:
$T \in \mathcal{B}(X)$ is said to satisfy a-Weyl's theorem if $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi^a_{00}(T)$
$T \in \mathcal{B}(X)$ is said to satisfy property (ω) if $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$.
We have [3]: property (ω) $\Longrightarrow$ a-Browder's theorem .
a-Weyl's theorem $\Longrightarrow$ Weyl's theorem, and property (ω)) $\Longrightarrow$ Weyl's theorem.
The next theorem [2] establishes the precise relationship between property (ω) and a-Browder's theorem:

Theorem 1.3. *[2, Theorem(2-7)].*
If $T \in \mathcal{B}(X)$ the following statements are equivalent
1) T satisfies property(ω)
2) a-Browder's theorem hold for T and $p^a_{00}(T) = \pi_{00}(T)$

In presence of SVEP we have the equivalence between Weyl's theorem and property (ω).

Theorem 1.4. *[2, Theorem(2-16)].*
Let $T \in \mathcal{B}(X)$ then the following equivalences hold.
1) if $T^\star$ has SVEP the property(w) holds for T if and only if Weyl's theorem holds for T, and this is the case if and only if a-Weyl's theorem holds for T
2) If T has SVEP:
$T^\star$ has property(ω) $\Longleftrightarrow$ $T^\star$ satisfies Weyl's theorem $\Longleftrightarrow$ $T^\star$ satisfies a-weyl's theorem

A result of Cao [5] shows the relations between hypercyclic (supercyclic) operators and the operators for which Weyl's theorem holds . More recently Duggel in[7] have given necessary and sufficient conditions for hypercyclic and supercyclic operators to satisfy a-weyl's theorem.
In this paper we study the property(ω) for hypercyclic/supercyclic operators, we prove that if $T \in \mathcal{H}P(X) \bigcup \mathcal{S}P(X)$, then T obeys the property(ω) if and only if $\pi_{00}(T) = \pi_{00}(T^\star)$; Further if X is separable Hilbert space, in theorem (2-1)(see below) we characterized the norm-closure of hypercyclicty /supercyclicty for an operators which satisfies the property(ω).

2 Main results

We begin by the example.
Example:
Consider $X = \ell^2(\mathbb{N})$ and let $T : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ where
$x = (x_1, x_2, ...) \longrightarrow T(x) = (x_2/2, x_3/3, ..., x_n/n, ...)$
T is surjective quasi-nilpotent, $\overline{\bigcup_{k\geq 1} N(T^k)} = X$, $\sigma(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$, by corollary (1) T is supercyclic. Consider the Hilbert adjoint $T^{'}$ we have $T^{'}(x) = T^{'}(x_1, x_2, ...) = (0, x_1/2, x_2/3, ..., x_n/n+1, ...)$
$T^{'}$ is injective quasi-nilpotent, $\sigma(T^{'}) = \{0\}$ and, $\pi_{00}(T^{'}) = \emptyset$.
Or $\overline{\pi_\infty(T^{'})} = \pi_\infty(T^\star)$ where the barre design the conjugate complex (see proof of [4;Th(2-4)]), then $\pi_{00}(T^\star) = \emptyset$, consequently $\pi_{00}(T^\star) \neq \pi_{00}(T)$.
In the other hand T et $T^\star$ has SVEP because $\sigma(T) = \sigma(T^\star) = \{0\}$, by [3,Theorem(1-5)] we have $\sigma(T) = \sigma_a(T)$ and $\sigma_w(T) = \sigma_{uw}(T)$, $\sigma(T)\backslash\sigma_{uw}(T) = \emptyset$ because $\sigma(T) = \sigma_{uw}(T)$, and $\sigma_a(T)\backslash\sigma_{uw}(T) = \sigma(T)\backslash\sigma_w(T) \neq \pi_{00}(T)$.
Hence T does not satisfy property(ω).
In [11] Herrero proved that if $T \in \mathcal{H}P(X)$ then $\sigma_p(T^\star) = \emptyset$, and if $T \in \mathcal{S}P(X)$ then either $\sigma_p(T^\star) = \emptyset$ or $T = R \oplus \alpha I_{\mathbb{C}}$ where $\alpha \in \mathbb{C}\backslash\{0\}$ and $(1/\alpha)R \in \mathcal{H}P(X)$ (i.e $\sigma_p(T^\star) = \emptyset$ or $\sigma_p(T^\star) = \{\alpha\}$, $\alpha \neq 0$).
We give now the necessary and sufficient conditions for hypercyclic/supercyclic operators to have property (ω).

Proposition 2.1. *Let $T \in \mathcal{H}P(X) \bigcup \mathcal{S}P(X)$ then the following statement are equivalent.*
1) the property (ω) holds for T
2) $\pi_{00}(T) = \pi_{00}(T^\star)$

Proof: We show first that $T^\star$ satisfies weyl's theorem .
In fact, if $T \in \mathcal{H}P(X) \bigcup \mathcal{S}P(X)$ we have $\sigma_p(T^\star) = \emptyset$ or $\sigma_p(T^\star) = \{\alpha\}$ for $\alpha \in \mathbb{C}\backslash\{0\}$, this implies that $T^\star$ has SVEP, consequently T and $T^\star$ satisfies a-Browder's theorem (in particulary Browder's theorem).
-If $\sigma_p(T^\star) = \emptyset$ we have $\pi_{00}(T^\star) = \emptyset$, and from $p_{00}(T^\star) \subseteq \pi_{00}(T^\star)$ it then follows that $p_{00}(T^\star) = \pi_{00}(T^\star) = \emptyset$, since $T^\star$ has Browder's theorem it then by [2-Theorem(2-16)] $T^\star$ satisfies Weyl's theorem.
-If $\sigma_p(T^\star) = \{\alpha\}$ for some $\alpha \neq 0$ such that $\alpha \notin \sigma_b(T^\star)$ then $\pi_{00}(T^\star) = \{\alpha\}$ and $\sigma(T^\star)\backslash\sigma_b(T^\star) = p_{00}(T^\star) = \{\alpha\}$, consequently $p_{00}(T^\star) = \pi_{00}(T^\star) = \{\alpha\}$ then either by [2-Th(2-16)] $T^\star$ satisfies Weyl's theorem.
It should be noted that $p_{00}(T) = \sigma(T)\backslash\sigma_b(T) = \sigma(T)\backslash\sigma_w(T)$ because T satisfies a-Browder's theorem ($\sigma_{ub}(T) = \sigma_{uw}(T)$ and $\sigma_b(T) = \sigma_w(T)$), since $T^\star$ have SVEP then by [3-Th(1-5)] $\sigma(T) = \sigma_a(T)$ and $\sigma_w(T) = \sigma_{uw}(T)$, we then conclude that $p_{00}(T) = \sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}^a(T) \quad (\star)$.

We prove 2)$\Longrightarrow$1). Suppose that $\pi_{00}(T)=\pi_{00}(T^\star)$, since $T^\star$ satisfies Weyl's theorem then $p_{00}(T^\star)=\pi_{00}(T^\star)$ and by $(\star)$ $\pi_{00}(T^\star)=p_{00}(T^\star)=p_{00}(T)=p^a_{00}(T)$ therefore $p^a_{00}(T)=\pi_{00}(T)$, and since T have a-Browder's theorem we conclude by theorem (1-3) that T satisfies property(ω).
To show the opposite implication 1)$\Longrightarrow$2): suppose that T satisfies property(ω), then $p^a_{00}(T)=\pi_{00}(T)$, since $T^\star$ has Weyl's theorem and $(\star)$ we have $\pi_{00}(T^\star)=p_{00}(T^\star)=p_{00}(T)=p^a_{00}(T)=\pi_{00}(T)$ □

Corollory 2:
Suppose that for $T\in\mathcal{H}P(X)\bigcup\mathcal{S}P(X)$, T have $\pi_{00}(T)=\pi_{00}(T^\star)$ then T satisfies a-Weyl's theorem.

Proof: Since $\pi_{00}(T)=\pi_{00}(T^\star)$ and $T\in\mathcal{H}P(X)\bigcup\mathcal{S}P(X)$ then by proposition(2-1) T satisfies the property(ω), or $T^\star$ have SVEP , then by theorem(1-4) T satisfies a-Weyl's theorem. □

Corollory 3:
Suppose that X is separable and $T\in\mathcal{B}(X)$, if $\lambda I-T$ is surjective and $\bigcup_{k\geq 0}N(\lambda I-T)^k$ is dense for some λ, then $f(T)$ satisfies the property(ω) whenever $f\in\mathcal{H}_c(\sigma(T))$.

Proof:without loss of generality we assume $\lambda=0$. We have : $\overline{\bigcup_{k\geq 0}N(T^k)}=X$ and T is surjective, by corollary (1) $f(T)\in\mathcal{S}P(X)$ for $f\in\mathcal{H}_c(\sigma(T))$, on the other hand by [12-proposition3] $\sigma(T)$ is connected, since $\sigma(f(T)^\star)=\sigma(f(T))=f(\sigma(T))$ then $\sigma(f(T))$ and $\sigma(f(T)^\star)$ are connected. This implies that $\pi_{00}(f(T))=\pi_{00}(f(T)^\star)=\emptyset$, we conclude by proposition (2-1) that $f(T)$ satisfies the property(ω). □

In the case that X is the separable Hilbert space then for $T\in\mathcal{B}(X)$ satisfies the property(ω), we have the following result.

Theorem 2.1. .
Let $T\in\mathcal{B}(X)$ where X is separable Hilbert space, and suppose that T satisfies property(ω), then the following statements are equivalent:
1) $T\in\overline{\mathcal{H}P(X)}\iff$ 2) $\sigma(T)\bigcup\Gamma$ is connected
3) $T\in\overline{\mathcal{S}P(X)}\iff$ 4) $\sigma(T)\bigcup\Gamma_r$ is connected for some $r\geq 0$

Proof: 1)$\Longrightarrow$2) Suppose that $T\in\overline{\mathcal{H}P(X)}$,by theorem(1-1) $\sigma_w(T)\bigcup\Gamma$

is connected, $\sigma(T)=\sigma_b(T)$ and $ind(\lambda I-T)\geq 0$, $\forall\lambda\in\rho_{sF}(T)$. Since T satisfies the property(ω), T satisfies Browder's theorem. Hence
$\sigma_b(T)=\sigma_w(T)$ and $\sigma(T)=\sigma_b(T)=\sigma_w(T)$ so $\sigma(T)\bigcup\Gamma$ is connected.
2)$\Longrightarrow$1). Now suppose that $\sigma(T)\bigcup\Gamma$ is connected . First we claim that $\sigma(T)=\sigma_w(T)$, to see this suppose that there is $\lambda_0\in\sigma(T)\backslash\sigma_w(T)$.
Observe that T satisfies Browder's theorem, consequently $\sigma_w(T)=\sigma_b(T)$ and $\sigma(T)\backslash\sigma_w(T)=\sigma(T)\backslash\sigma_b(T)=p_{00}(T)\subseteq\pi_{00}(T)$.
Hence $\lambda_0\in\pi_{00}(T)$ and $\lambda_0\in$ iso $\sigma(T)$, this entails that there exists an open disc $\mathcal{D}(\lambda_0,\varepsilon)$ such that $\mathcal{D}(\lambda_0,\varepsilon)\bigcap\sigma(T)=\{\lambda_0\}$, we have two cases : $|\lambda_0|=1$ or $|\lambda_0|\neq 1$.
-If $|\lambda_0|\neq 1$ then (we can take ε small) we have $\mathcal{D}(\lambda_0,\varepsilon)\bigcap\Gamma=\emptyset$ and since $(\mathcal{D}(\lambda_0,\varepsilon)\bigcap\sigma(T))\bigcup(\mathcal{D}(\lambda_0,\varepsilon)\bigcap\Gamma)=\mathcal{D}(\lambda_0,\varepsilon)\bigcap(\sigma(T)\bigcup\Gamma)=\{\lambda_0\}$ and $\sigma(T)\bigcup\Gamma$ is connected, we have $\sigma(T)\bigcup\Gamma)=\{\lambda_0\}$, this is impossible.
-If $|\lambda_0|=1$, then $\mathcal{D}(\lambda_0,\varepsilon)\bigcap\Gamma=\{z\in\Gamma:\theta_0<arg(z)<\theta_1\ \ ,\theta_0,\theta_1\in IR\}=\Gamma_{\theta_0,\theta_1}$, we have:
$(\mathcal{D}(\lambda_0,\varepsilon)\bigcap\sigma(T))\bigcup(\mathcal{D}(\lambda_0,\varepsilon)\bigcap\Gamma)=\mathcal{D}(\lambda_0,\varepsilon)\bigcap(\sigma(T)\bigcup\Gamma)=\{\lambda_0\}\bigcup\Gamma_{\theta_0,\theta_1}=\Gamma_{\theta_0,\theta_1}$; Since $\sigma(T)\bigcup\Gamma$ is connected we have $\sigma(T)\bigcup\Gamma=\Gamma_{\theta_0,\theta_1}$, and this is contradiction because $\Gamma_{\theta_0,\theta_1}\subsetneqq\Gamma$.
We then conclude that $\pi_{00}(T)=\emptyset$, $\sigma(T)=\sigma_w(T)$ and $\sigma_w(T)\bigcup\Gamma$ is connected.
We show that $ind(\lambda I-T)\geq 0\ \forall\lambda\in\rho_{sF}(T)$, in fact suppose there exists $\lambda_0\in\rho_{sF}(T)$ such that $ind(\lambda_0 I-T)<0$, Hence $\lambda_0\in\sigma_w(T)=\sigma_b(T)=\sigma(T)$
From $ind(\lambda_0 I-T)<0$ and $\lambda_0\in\rho_{sF}(T)$, we have that
$\alpha(\lambda_0 I-T)<\beta(\lambda_0 I-T)$ and $(\lambda_0 I-T)\in\Phi_+(X)$, two cases are present:
$\alpha(\lambda_0 I-T)=0$ or $\alpha(\lambda_0 I-T)>0$.
-First case : if $\alpha(\lambda_0 I-T)=0$, then $\lambda_0 I-T$ is injective, since $(\lambda_0 I-T)(X)$ is closed, then $\lambda_0\notin\sigma_a(T)$ and $\lambda_0\in\sigma(T)\backslash\sigma_a(T)$. By [1-corollary(2-50)], T have SVEP in λ_0 and since $(\lambda_0 I-T)\in\Phi_+(X)$. This implies that
$\lambda_0\in$ iso $\sigma_a(T)\subseteq\sigma_a(T)$, a contradiction.
-Second case: $0<\alpha(\lambda_0 I-T)<\beta(\lambda_0 I-T)$, then $\lambda_0\in\sigma_a(T)$, on the other hand T satisfies a-Browder's theorem and $p^a_{00}(T)=\pi_{00}(T)$, then
$\sigma_{uw}(T)=\sigma_{ub}(T)$ and $\sigma_a(T)\backslash\sigma_{uw}(T)=\sigma_a(T)\backslash\sigma_{ub}(T)=p^a_{00}(T)=\pi_{00}(T)=\emptyset$.
From this it then follows that $\sigma_a(T)=\sigma_{uw}(T)$, since $\lambda_0\in\sigma_a(T)$ then $\lambda_0\in\sigma_{uw}(T)$, this contradicts $(\lambda_0 I-T)\in\Phi_+(X)$ and $ind(\lambda_0 I-T)<0$.
We conclude that $ind(\lambda_0 I-T)\geq 0$, $\forall\lambda\in\rho_{sF}(T)$.
Finally, we have: $\sigma_w(T)\bigcup\Gamma$ is connected , $\sigma(T)=\sigma_b(T)=\sigma_w(T)$ and $ind(\lambda_0 I-T)\geq 0\quad\forall\lambda\in\rho_{sF}(T)$, by theorem(1-1) $T\in\overline{\mathcal{H}P(X)}$.
Similarly by theorem(1-2) we can proved 3)$\Longleftrightarrow$4) □

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Received: November, 2011