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Trigonometric Approximation of Signals (Functions) in L_p-Norm

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Abstract

Broadly speaking, Signals are treated as functions of one variable and images are represented by functions of two variables. The study of these concepts is directly related to the emerging area of information technology. In this paper, we extend two theorems of Leindler [J. Math. Anal. Appl. 302 (2005) 129-136], where he has taken less stringent conditions on the generating sequence $\{p_n\}$ given by Chandra [J. Math. Anal. Appl. 275, 2002, 13-26], to more general classes of triangular matrix methods. Our Theorems also generalize theorem 4 partially and (ii) part of theorem 5 of M.L. Mittal, B.E. Rhoades, V.N. Mishra, U. Singh [J. Math. Anal. Appl. 326(2007) 667-676] by dropping the monotonicity on the elements of matrix rows.

1. Introduction

Let f be a 2π-periodic signal (function) and let $f \in L_p[0,2\pi] = L_p$ for $p \geq 1$. Then the Fourier series of function (signal) f at any point x is given by

$$f(x) \simeq \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos kx + b_k \sin kx\right) \equiv \sum_{k=0}^{\infty} u_k(f;x), \quad (1.1)$$

with partial sums $s_n(f;x)$ -a trigonometric polynomial of degree(or order) n, of the first $(n+1)$ terms.

We define

$$\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), \qquad \forall n \geq 0, \qquad (1.2)$$

where $T \equiv (a_{n,k})$ is a linear operator represented by the infinite lower triangular matrix. The series (1.2) is said to be T – summable to s, if $\tau_n(f;x) \to s$ as $n \to \infty$.

The T – operator reduces to the Nörlund (N_p) -operator, if $a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $p_{-1} = 0 = P_{-1}$. In this case, the transform $\tau_n(f;x)$ reduces to the Nörlund transform $N_n(f;x)$.

The T – operator reduces to the weighted (Riesz) $(\bar{N}_p)$ - operator, if $a_{n,k} = \begin{cases} p_k/P_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $p_{-1} = 0 = P_{-1}$. In this case, the transform $\tau_n(f;x)$ reduces to the weighted (Riesz) transform $\bar{N}_n(f;x)$ (or $R_n(f;x)$).

A signal (function) $f \in Lip\,\alpha$, for $0 < \alpha \le 1$, if $|f(x+t) - f(x)| = O(t^\alpha)$.

A signal (function) $f \in Lip(\alpha, p)$ for $p \ge 1$, $0 < \alpha \le 1$, if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p\, dx \right\}^{1/p} = O(t^\alpha).$$

The integral modulus of continuity of function $f \in L_p[0, 2\pi]$ is defined by

$$\omega_p(\delta; f) = \sup_{0<|h|\le\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p\, dx \right\}^{1/p}.$$

If, for $\alpha > 0$, $\omega_p(\delta; f) = O(\delta^\alpha)$, then $f \in Lip(\alpha, p)$ $(p \ge 1)$.

Throughout $\|.\|_p$ will denote L_p-norm with respect to x and will be defined by

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p\, dx \right\}^{1/p} \qquad \left(f \in L_p\,(p \ge 1)\right).$$

Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative entries. A matrix T is said to have monotone rows if, for each n, $\{a_{n,k}\}$ is either non-increasing or non-decreasing in k, $0 \le k \le n$.

A lower triangular matrix T is called a hump matrix if, for each n, there exists an integer $k_0 = k_0(n)$, such that $a_{n,k} \le a_{n,k+1}$ for $0 \le k < k_0$, and $a_{n,k} \ge a_{n,k+1}$ for $k_0 \le k < n$.

A positive sequence $c := \{c_n\}$ is called almost monotone decreasing (or increasing) if there exists a constant $K := K(c)$, depending on the sequence c only, such that for all $n \ge m$, $c_n \le K c_m$ $(K c_n \ge c_m)$.

Such sequences will be denoted by $c \in AMDS$ and $c \in AMIS$, respectively.

We write $D_n(t) = \left(\sin\left(n + 1/2\right)t\right) / \left(2\sin\left(t/2\right)\right)$, is known as Dirichlet Kernel of degree (or order) n.

$$s_n(f)=\frac{1}{\pi}\int_0^{2\pi} f(x+t)\,D_n(t)\,dt, \qquad \sigma_n(f;x)=\frac{1}{n+1}\sum_{m=0}^{n} s_m(f;x),$$

$$N_n(f;x)=\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_k(f;x), \qquad P_n=\sum_{r=0}^{n} p_r \neq 0,\ p_{-1}=0=P_{-1}.$$

$$R_n(f;x)=\frac{1}{P_n}\sum_{k=0}^{n} p_k s_k(f;x), \qquad P_n=\sum_{r=0}^{n} p_r \neq 0,\ p_{-1}=0=P_{-1}.$$

$$A_{n,k}=\sum_{r=k}^{n} a_{n,r}, \quad t_n=\sum_{k=0}^{n} a_{n,k}=A_{n,0}, \quad b_{n,k}=\frac{A_{n,k}-A_{n,0}}{k} \quad \forall\ 1\le k\le n,$$

$\Delta_k a_{n,k}=a_{n,k}-a_{n,k+1}$, [x]- denotes the greatest integer contained in x.

In this paper, the signals (functions) f are approximated by trigonometric polynomials τ_n of order (or degree) n and the degree of approximation $E_n(f)$ is given by $E_n(f)=\underset{n}{\mathrm{Min}}\left\| f(x)-\tau_n(f;x)\right\|_p$,

in terms of n. This method of approximation is called trigonometric Fourier Approximation (tfa) and used in the theory of Machines in Mechanical Engineering.

Let $\sigma_n(f)$ denote the nth term of the (C, 1) transform of the partial sums of the Fourier series of a 2π-periodic function f. In 1937, Quade [1] has extended the results for the functions $f\in \mathrm{Lip}(\alpha, p)$ for $0<\alpha\le 1$, on Cesàro matrix:

Theorem 1 [1]. If $f\in \mathrm{Lip}(\alpha, p)$ for $0<\alpha\le 1$, then

$$\left\| f(x)-\sigma_n(f;x)\right\|_p = O(n^{-\alpha}) \tag{1.4}$$

for either (i) $p>1$ and $0<\alpha\le 1$ or (ii) $p=1$ and $0<\alpha<1$. and if $p=\alpha=1$, then

$$\left\| f(x)-\sigma_n(f;x)\right\|_1 = O\left(n^{-1}\log(n+1)\right). \tag{1.5}$$

In a recent paper Chandra [4] extended the work of Quade [1] and proved the following theorems, where $N_n(f)$ and $R_n(f)$ denote the nth terms of the Nörlund and weighted mean transforms of the sequences of partial sums, respectively.

Theorem 2 [4]. Let $f\in \mathrm{Lip}(\alpha,p)$ and let $\{p_n\}$ be a positive sequence such that

$$(n+1)\,p_n = O\,(P_n). \tag{1.6}$$

If either

(i) $p>1,\ 0<\alpha\le 1$ and (ii) $\{p_n\}$ is monotonic, or (i) $p=1,\ 0<\alpha<1$ and (ii) $\{p_n\}$ is non-decreasing sequence, then

$$\left\| f(x)-N_n(f;x)\right\|_p = O(n^{-\alpha}). \tag{1.7}$$

Theorem 3 [4]. Let $f\in \mathrm{Lip}(\alpha,p)$ and let $\{p_n\}$ be positive. Suppose that either

(i) $p>1,\ 0<\alpha\le 1,$ and (ii) $\displaystyle\sum_{k=0}^{n-1}\left|\Delta\left(\frac{P_k}{k+1}\right)\right| = O\left(\frac{P_n}{n+1}\right)$, or

(i) $p=1,\ 0<\alpha<1$ and (ii) $\{p_n\}$ with (1.6) is positive and non-decreasing. Then

$$\left\| f(x)-R_n(f;x)\right\|_p = O\,(n^{-\alpha}). \tag{1.8}$$

Theorem 4 [4]. Let $f \in Lip(1,1)$ and let $\{p_n\}$ with (1.6) be positive and that

$$(n+1)^{-\eta} p_n \text{ be non-decreasing for some } \eta > 0. \tag{1.9}$$

Then $$\| f(x) - R_n(f;x) \|_1 = O(n^{-1}). \tag{1.10}$$

These results of Chandra [4], as mentioned by him, are sharper and new and hence are interesting (in view of Leindler [2, p.130] also).

Recently, Leindler [2] has extended theorem 2 and theorem 3 by taking less stringent conditions on the generating sequence $\{p_n\}$. He proved:

Theorem 5 [2]. Let $f \in Lip(\alpha, p)$ and let $\{p_n\}$ be positive. If one of the conditions

(i) $p > 1,\ 0 < \alpha < 1$ and $\{p_n\} \in AMDS$,

(ii) $p > 1,\ 0 < \alpha < 1,\ \{p_n\} \in AMIS$ and (1.6) holds,

(iii) $p > 1,\ \alpha = 1$ and $\sum_{k=1}^{n-1} k |\Delta p_k| = O(P_n)$,

(iv) $p > 1,\ \alpha = 1,\ \sum_{k=0}^{n-1} |\Delta p_k| = O(P_n / n)$ and (1.6) holds,

(v) $p = 1,\ 0 < \alpha < 1,\ \sum_{k=-1}^{n-1} |\Delta p_k| = O(P_n / n)$ maintains, then

$$\| f(x) - N_n(f;x) \|_p = O(n^{-\alpha}).$$

Theorem 6 [2]. Let $f \in Lip(\alpha, 1), 0 < \alpha < 1$. If the positive sequence $\{p_n\}$ satisfies (1.6) and the condition $\sum_{k=0}^{n-1} |\Delta p_k| = O(P_n / n)$ holds, then

$$\| f(x) - R_n(f;x) \|_1 = O(n^{-\alpha}). \tag{1.11}$$

Very recently M.L. Mittal, B.E. Rhoades, V.N. Mishra, U. Singh [3] have generalized two theorems 2 and 3 (Chandra [4, Theorems 1 and 2]) to more general classes of triangular matrix methods. They prove:

Theorem 7 [3]. Let $f \in Lip(\alpha, p)$ and let T have monotone rows and satisfy

$$| t_n - 1 | = O(n^{-\alpha}). \tag{1.12}$$

(i) If $p > 1,\ 0 < \alpha < 1,$ and T also satisfies

$$(n+1) \max\{a_{n,0}, a_{n,r}\} = O(1), \tag{1.13}$$

where $r := [n/2]$, then $\| \tau_n(f;x) - f(x) \|_p = O(n^{-\alpha}).$ (1.14)

(ii) If $p > 1,\ \alpha = 1,$ then (1.14) is satisfied.

(iii) If $p = 1,\ 0 < \alpha < 1,$ and T also satisfies

$$(n+1) \max\{a_{n,0}, a_{n,n}\} = O(1), \tag{1.15}$$

then (1.14) is satisfied.

Theorem 8 [3]. Let T be a hump matrix satisfying condition (1.12) and

$$(n+1)\max_{k}\left\{ a_{n,k} \right\} = O(1). \qquad (1.16)$$

Then, if either

(i) $p > 1, 0 < \alpha < 1$, or (ii) $p = 1, 0 < \alpha < 1$, condition (1.14) is satisfied. In this paper, we generalize two theorems 5 and 6 of Leindler [2, Theorems 1 and 2] to more general classes of triangular matrix methods. Our Theorems also generalize partially theorem 7 and (ii) part of theorem 8 of M.L. Mittal, B.E. Rhoades, V.N. Mishra, U. Singh [3] respectively by dropping the monotonicity on the elements of matrix rows. We prove:

Theorem 9. Let $f \in Lip(\alpha, p)$, and let $T = (a_{n,k})$ be an infinite triangular matrix with positive entries $(a_{n,k})$ with row sums 1 such that

$$(n+1)\, a_{n,0} = O(1). \qquad (1.17)$$

If one of the conditions

(i) $p > 1,\ 0 < \alpha < 1,\ \left\{a_{n,k}\right\} \in AMDS$ and (1.17) holds,

(ii) $p > 1,\ 0 < \alpha < 1,\ \left\{a_{n,k}\right\} \in AMIS$ and (1.17) holds,

(iii) $p > 1, \alpha = 1$ and $\sum_{k=0}^{n-1} (n-k) \left| \Delta_k a_{n,k} \right| = O(1)$, (1.18)

(iv) $p > 1, \alpha = 1, \sum_{k=0}^{n-1} \left| \Delta_k a_{n,k} \right| = O(a_{n,0})$ and (1.17) holds, (1.19)

(v) $p = 1,\ 0 < \alpha < 1,\ \sum_{k=0}^{n} \left| \Delta_k a_{n,k} \right| = O(a_{n,0})$ *and* (1.17) *holds,* (1.20)

maintains, then (1.14) is satisfied.

Theorem 10. Let $f \in Lip(\alpha, 1),\ 0 < \alpha < 1$. If the positive sequence $\left\{a_{n,k}\right\}$ satisfies condition (1.20), *then* $\| \tau_n(f;x) - f(x) \|_1 = O(n^{-\alpha})$. (1.21)

We note that:

(1) In case of Nörlund (N_p) - transform, condition (1.17) reduces to (1.6), while conditions (1.18) and (1.19) reduce to conditions (iii) and (iv) of theorem 5 respectively. Thus our Theorem 9 generalizes theorem 5. Similarly, in case of Riesz (R_p) - transform, Theorem 10 extends theorem 6.

(2) Further, it is easy to examine that the conditions of Theorems 9 and 10 claim less than the requirements of our Theorems 7 and 8 respectively for $A_{n,0} = 1 = t_n$. For example, the condition on the sum in (1.18) is always satisfied if the sequence $\left\{a_{n,k}\right\}$ is non-decreasing in k, then using (1.17), we get

$$\sum_{k=0}^{n-1}(n-k)\left| \Delta_k a_{n,k} \right| = \sum_{k=0}^{n-1}(n-k)\left| a_{n,k} - a_{n,k+1} \right| = \sum_{k=0}^{n-1}(n-k)(a_{n,k+1} - a_{n,k})$$
$$= A_{n,0} - (n+1)a_{n,0} = 1 + O(1) = O(1).$$

If $\{a_{n,k}\}$ is non-increasing in k and (1.17) holds then

$$\sum_{k=0}^{n-1}\left|\Delta_k a_{n,k}\right|=\sum_{k=0}^{n-1}\left|a_{n,k}-a_{n,k+1}\right|=\sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})=a_{n,0}-a_{n,n}\le a_{n,0}=O(n^{-1})$$

is also true.

2. Lemmas In order to prove our Theorems 9 and 10, we require the following lemmas.

Lemma 1 [1]. If $f\in Lip(\alpha,1),\ 0<\alpha<1,$ then

$$\|f(x)-\sigma_n(f;x)\|_1=O(n^{-\alpha}). \tag{2.1}$$

Lemma 2 [1]. If $f\in Lip(1,p),\ p>1,$ then

$$\|\sigma_n(f;x)-s_n(f;x)\|_p=O(n^{-1}). \tag{2.2}$$

Lemma 3 [1]. Let, for $0<\alpha\le 1$ and $p>1,\ f\in Lip(\alpha,p).$ Then

$$\|f(x)-s_n(f;x)\|_p=O(n^{-\alpha}). \tag{2.3}$$

Lemma 4 [3]. Let T have monotone rows and satisfy (1.13). Then, for $0<\alpha<1$,

$$\sum_{k=0}^{n}a_{n,k}(k+1)^{-\alpha}=O(n^{-\alpha}). \tag{2.4}$$

3. Proof of the Theorems

3.1 Proof of Theorem 9.

Cases I and II. If $p>1,\ 0<\alpha<1$, we prove the cases (i) and (ii) simultaneously. The proof runs similar to the case (i) of Theorem 7, dropping the second term as $t_n=1=A_{n,0},\ \forall n$. Let $\{a_{n,k}\}$ be either AMDS or AMIS. Thus for the sake of completeness, we have

$$\tau_n(f;x)-f(x)=\sum_{k=0}^{n}a_{n,k}s_k(f;x)-f(x)=\sum_{k=0}^{n}a_{n,k}\left(s_k(f;x)-f(x)\right), \tag{3.1}$$

thus $\|\tau_n(f;x)-f(x)\|_p\le\sum_{k=0}^{n}a_{n,k}\|s_k(f;x)-f(x)\|_p=\sum_{k=0}^{n}a_{n,k}O(k^{-\alpha})=O(n^{-\alpha}),$

in view of Lemmas (3) and (4). Next, we consider the case (iv).

Case IV. If $p>1,\ \alpha=1,$ we have

$$\tau_n(f;x)-f(x)=\tau_n(f;x)-s_n(f;x)+s_n(f;x)-f(x).$$

Now, using Lemma 3, we get

$$\begin{aligned}\|\tau_n(f;x)-f(x)\|_p&\le\|\tau_n(f;x)-s_n(f;x)\|_p+\|s_n(f;x)-f(x)\|_p\\&=\|\tau_n(f;x)-s_n(f;x)\|_p+O(n^{-1}).\end{aligned} \tag{3.2}$$

Now to prove our theorem, it remains to show that

$$\|\tau_n(f;x)-s_n(f;x)\|_p=O(n^{-1}). \tag{3.3}$$

Now, we write $\tau_n(f;x)=\sum_{k=0}^{n}a_{n,k}s_k(f;x)=\sum_{k=0}^{n}a_{n,k}\left(\sum_{i=0}^{k}u_i(f;x)\right)=\sum_{k=0}^{n}A_{n,k}u_k(f;x)$,

and thus, as $A_{n,0}=1$, we have

$$\tau_n(f;x)-s_n(f;x)=\sum_{k=0}^{n}A_{n,k}u_k(f;x)-\sum_{k=0}^{n}u_k(f;x).1$$

$$=\sum_{k=1}^{n}\left(\frac{A_{n,k}-A_{n,0}}{k}\right)k\,u_k(f;x)=\sum_{k=1}^{n}b_{n,k}\,k\,u_k(f;x).$$

Hence by Abel's transformation, we obtain

$$\tau_n(f;x)-s_n(f;x)=\sum_{k=1}^{n-1}\left(\Delta_k\, b_{n,k}\right)\left(\sum_{j=1}^{k}ju_j(f;x)\right)+b_{n,n}\sum_{j=1}^{n}ju_j(f;x).$$

Thus by triangle inequality, we find

$$\|\tau_n(f;x)-s_n(f;x)\|_p\le\sum_{k=1}^{n-1}\left|\Delta_k b_{n,k}\right|\left\|\sum_{j=1}^{k}ju_j(f;x)\right\|_p+\left|b_{n,n}\right|\left\|\sum_{j=1}^{n}ju_j(f;x)\right\|_p. \quad (3.4)$$

Now $$\sigma_n(f;x)-s_n(f;x)=\frac{1}{n+1}\sum_{m=0}^{n}s_m(f;x)-s_n(f;x)$$

$$=\frac{1}{n+1}\sum_{m=0}^{n}s_m(f;x)-\sum_{k=0}^{n}u_k(f;x)=-\frac{1}{n+1}\sum_{j=1}^{n}ju_j(f;x).$$

Therefore by Lemma 2, we have

$$\left\|\sum_{j=1}^{n}ju_j(f;x)\right\|_p=(n+1)\|\sigma_n(f;x)-s_n(f;x)\|_p=(n+1)O(n^{-1})=O(1). \quad (3.5)$$

We note that

$$\left|b_{n,n}\right|=\left|\frac{A_{n,n}-A_{n,0}}{n}\right|=\frac{\left|A_{n,0}-A_{n,n}\right|}{n}=\frac{\left(A_{n,0}-A_{n,n}\right)}{n}=\frac{A_{n,0}}{n}=O(1/n).$$

Thus $$\left\|b_{n,n}\sum_{j=1}^{n}ju_j(f;x)\right\|_p=\left|b_{n,n}\right|\left\|\sum_{j=1}^{n}ju_j(f;x)\right\|_p=O(n^{-1}). \quad (3.6)$$

As in case (ii) of proof of Theorem 7, we write

$$\Delta_k b_{n,k}=\frac{1}{k(k+1)}\left[(k+1)a_{n,k}-\sum_{r=0}^{k}a_{n,r}\right]. \quad (3.7)$$

Next we shall verify by mathematical induction that

$$\left|\sum_{r=0}^{k}a_{n,r}-(k+1)a_{n,k}\right|\le\sum_{r=0}^{k-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right|. \quad (3.8)$$

If $k=1$, then $$\left|\sum_{r=0}^{1}a_{n,r}-2a_{n,1}\right|=\left|a_{n,0}-a_{n,1}\right|.$$

Thus (3.8) holds. Now let us suppose that (3.8) holds for $k=m$ i.e.

$$\left|\sum_{r=0}^{m} a_{n,r}-(m+1)a_{n,k}\right| \le \sum_{r=0}^{m-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right|, \tag{3.9}$$

and we have to show that (3.8) is true for $k=m+1$.
For $k=(m+1)$ and using (3.9), we get

$$\left|\sum_{r=0}^{m+1} a_{n,r}-(m+2)a_{n,m+1}\right| = \left|\sum_{r=0}^{m} a_{n,r}-(m+1)a_{n,m+1}\right|$$

$$=\left|\sum_{r=0}^{m} a_{n,r}-(m+1)a_{n,m}+(m+1)a_{n,m}-(m+1)a_{n,m+1}\right|$$

$$\le \sum_{r=0}^{m-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right|+(m+1)\left|a_{n,m}-a_{n,m+1}\right| = \sum_{r=0}^{(m+1)-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right|,$$

which shows that (3.8) is true for $k=m+1$. Thus (3.8) holds $\forall\ k\in N$.
Using (1.17), (3.7) and (3.8), we find

$$\sum_{k=1}^{n}\left|\Delta_k b_{n,k}\right| = \sum_{k=1}^{n}\left|\Delta_k\left\{k^{-1}(A_{n,k}-A_{n,0}\right\}\right| = \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\left|(k+1)a_{n,k}-\sum_{r=0}^{k}a_{n,r}\right|$$

$$=\sum_{k=1}^{n}k^{-1}(k+1)^{-1}\left|\sum_{r=0}^{k}a_{n,r}-(k+1)a_{n,k}\right|$$

$$\le\sum_{k=1}^{n}k^{-1}(k+1)^{-1}\sum_{r=0}^{k-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right| = \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|$$

$$\le\sum_{m=1}^{n}m\left|\Delta_m a_{n,m-1}\right|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} = \sum_{k=0}^{n-1}\left|\Delta_k a_{n,k}\right| = O(a_{n,0}) = O(n^{-1}). \tag{3.10}$$

Combining (3.4), (3.5), (3.6) and (3.10) yields (3.3). Consequently, using (3.3) from (3.2), we obtain $\quad \|\tau_n(f;x)-f(x)\|_p = O(n^{-1})$.

This completes the proof of case (iv).

Case III. If $p>1, \alpha=1$. Now to prove this case first of all we prove that the condition $\sum_{k=0}^{n-1}(n-k)\left|\Delta_k a_{n,k}\right| = O(1)$ implies that

$$B_n \equiv \sum_{k=1}^{n}\left|\Delta_k b_{n,k}\right| = \sum_{k=1}^{n}\left|\Delta_k\left\{k^{-1}(A_{n,k}-A_{n,0}\right\}\right| = O(n^{-1}). \tag{3.11}$$

For this, using (3.8) as in case (iv), we have

$$B_n = \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\left|(k+1)a_{n,k}-\sum_{r=0}^{k}a_{n,r}\right| \le \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\sum_{r=0}^{k-1}(r+1)\left|a_{n,r}-a_{n,r+1}\right|$$

$$\le\left(\sum_{k=1}^{r}+\sum_{k=r}^{n}\right)k^{-1}(k+1)^{-1}\sum_{m=1}^{k}m\left|\Delta_m a_{n,m-1}\right| = B_1+B_2, \text{ say.} \tag{3.12}$$

Now, using (1.18) and interchanging the order of summation, we get

$$B_1 \equiv \sum_{k=1}^{r} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m\left|\Delta_m a_{n,m-1}\right| \le \sum_{m=1}^{r} m\left|\Delta_m a_{n,m-1}\right| \sum_{k=m}^{\infty}\frac{1}{k(k+1)}$$

$$=\sum_{m=1}^{r}\left|\Delta_m a_{n,m-1}\right| = \sum_{m=n-r+1}^{n}\left|\Delta_{n-m}a_{n,n-m}\right| \le \sum_{m=r-1}^{n}\left|\Delta_{n-m}a_{n,n-m}\right|\left(\frac{m}{r-1}\right)$$

$$\le \frac{1}{r-1}\sum_{m=1}^{n} m\left|\Delta_{n-m}a_{n,n-m}\right| = \frac{1}{r-1}\sum_{k=0}^{n-1}(n-k)\left|\Delta_k a_{n,k}\right| = \frac{1}{r-1}O(1) = O(n^{-1}). \qquad (3.13)$$

On the other hand

$$B_2 \equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m\left|\Delta_m a_{n,m-1}\right| \le \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\left[\left(\sum_{m=1}^{r}+\sum_{m=r}^{k}\right) m\left|\Delta_m a_{n,m-1}\right|\right]$$

$$= B_{n,1}+B_{n,2}, \text{ say.} \qquad (3.14)$$

Now, using (1.18), we obtain

$$B_{n,1} \equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=1}^{r} m\left|\Delta_m a_{n,m-1}\right|$$

$$\le \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=1}^{r}\left|\Delta_m a_{n,m-1}\right| = \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=n-r+1}^{n}\left|\Delta_{n-m}a_{n,n-m}\right|$$

$$\le \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=r-2}^{n}\left|\Delta_{n-m}a_{n,n-m}\right|\frac{m}{r-2} \le \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}\sum_{m=1}^{n} m\left|\Delta_{n-m}a_{n,n-m}\right|$$

$$= \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}\sum_{k=0}^{n-1}(n-k)\left|\Delta_k a_{n,k}\right| = \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}O(1) = O(1/n), \qquad (3.15)$$

again using (1.18) and interchanging the order of summation, we have

$$B_{n,2} \equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=r}^{k} m\left|\Delta_m a_{n,m-1}\right| \le \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=r}^{k}\left|\Delta_m a_{n,m-1}\right|$$

$$\le \frac{1}{r+1}\sum_{m=r}^{n}\left|\Delta_m a_{n,m-1}\right|\sum_{k=m}^{n} 1 = \frac{1}{r+1}\sum_{m=r}^{n}(n-m+1)\left|\Delta_m a_{n,m-1}\right|$$

$$= \frac{1}{r+1}\sum_{k=r-1}^{n-1}(n-k)\left|\Delta_k a_{n,k}\right| = \frac{1}{r+1}O(1) = O(n^{-1}). \qquad (3.16)$$

From (3.12), (3.13), (3.14), (3.15) and (3.16), we get (3.11).
Thus (3.2), (3.4) and Lemma 3 again yield (1.14).

Case V. If $p=1, 0<\alpha<1$, using (3.1), $a_{n,n+1}=0$ and the Abel's transformation, we obtain

$$\tau_n(f;x)-f(x) = \sum_{k=0}^{n-1}(\Delta_k a_{n,k})\left\{\sum_{r=0}^{k}\left(s_r(f;x)-f(x)\right)\right\} + (a_{n,n}-a_{n,n+1})\sum_{r=0}^{n}\left(s_r(f;x)-f(x)\right)$$

$$= \sum_{k=0}^{n}(\Delta_k a_{n,k})\left\{\sum_{r=0}^{k}\left(s_r(f;x)-f(x)\right)\right\} = \sum_{k=0}^{n}(\Delta_k a_{n,k})(k+1)\left(\sigma_k(f;x)-f(x)\right).$$

Hence, by condition (1.17), (1.20) and Lemma 1, we find

$$\| \tau_n(f;x)-f(x) \|_1 \le \sum_{k=0}^{n}(k+1)\left| \Delta_k a_{n,k} \right| \| \sigma_k(f;x)-f(x) \|_1 = O\left\{ \sum_{k=0}^{n}(k+1)^{1-\alpha} \left| \Delta_k a_{n,k} \right| \right\}$$

$$= O(n^{1-\alpha}) \sum_{k=0}^{n} \left| \Delta_k a_{n,k} \right| = O(n^{1-\alpha}) O(a_{n,0}) = O(n^{1-\alpha}) O(n^{-1}) = O(n^{-\alpha}).$$

This completes the proof of case (v) and consequently the proof of Theorem 9 is complete.

3.2 Proof of Theorem 10

The proof is exactly the same as the proof of case (v) of Theorem 9.
If we write $a_{n,k} = p_k / P_n$ in the proof of case (v) of Theorem 9, we get the proof of theorem 6 and hence the proof of case (v) of Theorem 9 will be same as the proof of Theorem 10. □

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References

[1]. E.S. Quade, Trigonometric approximation in the mean, Duke Math. J. 3 (1937) 529-542.

[2]. L. Leindler, Trigonometric approximation in L_p-norm, J. Math. Anal. Appl. 302 (2005) 129-136.

[3]. M. L. Mittal, B. E. Rhoades, V. N. Mishra and Uaday Singh, Using infinite matrices to approximate functions of class Lip(α, p) using trigonometric polynomials, J. Math. Anal. Appl. 326 (2007) 667-676.

[4]. P. Chandra, Trigonometric approximation of functions in L_p-norm, J. Math. Anal. Appl. 275 (2002) 13-26.

[5]. Vishnu Narayan Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee, Roorkee - 247 667, Uttarakhand, India.

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