Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 20, 953 - 962

On Some New Identities for Ramanujan's Cubic Continued Fraction

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Abstract

In this paper, we establish some new modular relations connecting Ramanujan's cubic continued fraction $V(q)$ with $V(q^n)$, for $n = 4$, 6, 8, 10, 12, 14, 16 and 22.

Mathematics Subject Classification: 33D10, 11A55, 11F27

Keywords: Cubic continued fraction, Modular equations, Theta-function

1 Introduction

On page 366 of his 'lost' notebook [11], S. Ramanujan has recorded cubic continued fraction

$$V(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \qquad |q|<1 \tag{1}$$

and other identities related to $V(q)$. H. H. Chan [4] has established these identities. Subsequently, many mathematicians contributed to the theory of Ramanujan's cubic continued fraction. Some of them are N. D. Baruah [2], C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan [1], Mahadeva Naika [6], Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy [9], B. Cho, J. K. Koo and Y. K. Park [5], Mahadeva Naika, S. Chandankumar and Bairy [7], [8].

In this paper, we establish several new modular identities connecting $V(q)$ with $V(q^n)$, for n = 4, 6, 8, 10, 12, 14, 16 and 22.

2 Preliminary results

In Chapter 16, of his second notebook [10], [3, pp.257-262], Ramanujan develops the theory of theta-function and his theta-function is defined by

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \end{aligned}$$

where $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}), \ |q|<1$.

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{3}$$

Now, we define modular equation in brief. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{4}$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n, \quad 0 \le |z| < 1,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ for n a positive integer and a, b and c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{5}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{6}$$

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2} \quad \text{and} \quad n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{7}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (7). We say that β, γ and δ are of degrees n_1, n_2 and n_3, respectively over α. The multipliers m and m' are associated with α, β and γ, δ respectively.

We end this section by listing some relevant identities that are useful in proving our main results.

Lemma 2.1. *[3, Ch. 20, Entry 3 (xii), pp. 352–353] Let α, β and γ be of the first, third and ninth degrees respectively. Let m denote the multiplier connecting α, β and m' be the multiplier relating γ, δ, then*

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = -3\frac{m}{m'}, \tag{8}$$

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{9}$$

Lemma 2.2. *[3, Ch. 20, Entry 9 (vii), p. 377] We have*

$$\begin{aligned}&\left\{\psi(q^3)\psi(q^5)-q\psi(q)\psi(q^{15})\right\}\varphi(-q^3)\varphi(-q^5)\\&=\left\{\psi(q^3)\psi(q^5)+q\psi(q)\psi(q^{15})\right\}\varphi(-q)\varphi(-q^{15}).\end{aligned}\tag{10}$$

Lemma 2.3. *[3, Ch. 20, Entry 13 (i) and (ii), p. 401] Let α, β, γ and δ be of the first, third, seventh and twenty first degrees respectively. Let m denote the multiplier connecting α and β and m' be the multiplier relating γ and δ. Then*

$$\begin{aligned}\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4}&+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}\\&+4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6}=\frac{m}{m'},\end{aligned}\tag{11}$$

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4}&+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4}\\&+4\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/6}=\frac{m'}{m}.\end{aligned}\tag{12}$$

Lemma 2.4. *[3, Ch. 20, Entry 14 (i) and (ii), p. 408] Let α, β, γ and δ be of the first, third, eleventh and thirty third degrees respectively. Let m denote the multiplier connecting α and β and m' be the multiplier relating γ and δ. Then*

$$\begin{aligned}\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}&+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}-\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\\&-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12}=\sqrt{mm'},\end{aligned}\tag{13}$$

$$\begin{aligned}\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}&+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\\&-4\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12}=\frac{3}{\sqrt{mm'}}.\end{aligned}\tag{14}$$

Lemma 2.5. *[3, Ch. 17, Entry 10 (i), Entry 11 (ii), pp. 122–123] For $0<\alpha<1$,*

$$\varphi(q)=\sqrt{z},\tag{15}$$

$$\sqrt{2}q^{1/8}\psi(-q)=\sqrt{z}\{\alpha(1-\alpha)\}^{1/8},\tag{16}$$

where $z:={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$.

Lemma 2.6. *[4] If* $u = V(q)$ *and* $v = V(q^2)$*, then*

$$u^2 + 2v^2u - v = 0. \tag{17}$$

Lemma 2.7. *[6] We have*

$$\varphi^2(q) - \varphi^2(q^3) = 4q\chi^2(q)\psi(q^6)f(-q,-q^5), \tag{18}$$
$$\varphi^2(q) + \varphi^2(q^3) = 2\chi^2(q)\varphi(-q^3)f(q^2,q^4). \tag{19}$$

Lemma 2.8. *[1, Theorem 5.1] If* $P = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^3)}$*, then*

$$Q^4 + P^4Q^4 = 9 + P^4. \tag{20}$$

3 Modular identities for Ramanujan's cubic continued fraction

In this section, we establish several new modular relations connecting $V(q)$ with $V(q^n)$, for $n = 4$, 6, 8, 10, 12, 14, 16 and 22.

Theorem 3.1. *If* $a := V(q)$ *and* $b := V(q^6)$*, then*

$$\begin{aligned}&(16b^4 + 1 + 12b^2 + 4b + 16b^3)a^6 + 2b^4 - 3b^3 - b - b^2 - 4b^5\\ &+ (6b + 14b^2 + 12b^4 + 2b^3 + 8b^6 + 8b^5)a^3 = 0.\end{aligned} \tag{21}$$

Proof. Using the equations (8), (9), (15) and (16), we deduce that

$$d^2b_1^2 + a_1^2d^2 + 3c^2a_1^2 - b_1^2c^2 = 0, \tag{22}$$

where

$$c = \frac{\varphi(q)}{\varphi(q^3)},\ d = \frac{\varphi(q^3)}{\varphi(q^9)},\ a_1 = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)},\ b_1 = \frac{\psi(-q^3)}{q^{3/4}\psi(-q^9)}.$$

Using the equations (18) and (19), we deduce that

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} = \frac{1 + 2V(q)V(q^2)}{1 - 2V(q)V(q^2)}. \tag{23}$$

Using the equation (17), we deduce that

$$V(q^2) = \frac{3 + r}{1 - r}, \tag{24}$$

$$V(q) = \frac{1 + 2V(q^2)\{t - V^2(q^2)\}}{1 - 2V(q^2)\{t - V^2(q^2)\}}, \tag{25}$$

where $r := \pm\sqrt{1-8V^3(q)}$ and $t := \pm\sqrt{V^4(q^2)+V(q^2)}$.
Collecting the terms containing a_1^2 on one side of the equation (22), squaring and then employing the equations (20), (23), (24) and (25), we find that

$$\begin{aligned}&1-8bs-24b^4s-96b^7s-32b^3r-56b^6r+96b^9r-128b^{10}s\\&+128b^{12}r-6a^3+56b^4sr-32b^7sr-128b^{10}sr+128b^{12}+r\\&+56b^6+160b^9+128b^7sra^3+176b^4sra^3+24bsra^3+512b^9a^3\\&+464b^6a^3+1024b^9a^6+1024b^6a^6+144b^3a^6-240b^6ra^3\\&-128b^9ra^3-96b^3ra^3-208b^4sa^3-512b^7sa^3-1024b^7sa^6\\&+16b^3-2ra^3-16bsa^6+8bsr+40bsa^3-512b^4sa^6=0,\end{aligned} \tag{26}$$

where $s := \pm\sqrt{V^4(q^6)+V(q^6)}$.
Eliminating r and s from the equation (26), we obtain

$$\begin{aligned}&(2b^4-b^2-b-4b^5+12b^4a^3+6ba^3+16b^4a^6+4ba^6+2b^3a^3+8b^6a^3\\&+16b^3a^6+a^6+8b^5a^3+12b^2a^6+14b^2a^3-3b^3)(b^2-2b^4+7b^6-7b^5\\&+14b^7-8b^8+8b^9-8b^4a^3+186b^4a^6+ba^6+20b^3a^3-16b^9a^3+4b^6a^3\\&-88b^7a^3+72b^7a^6+a^{12}+44b^5a^3+88b^8a^3+29b^2a^6-12b^2a^3+144b^8a^6\\&+156b^5a^6+32b^{11}a^3-b^3-64b^{11}a^6+32b^5a^9+160b^8a^9+4b^2a^{12}+34b^2a^9\\&+256b^8a^{12}-64b^5a^{12}+64b^{12}a^6+128b^9a^9-320b^6a^9+64b^6a^{12}-16b^3a^{12}\\&-124b^3a^9-80b^{10}a^3-32a^6b^{10}+220a^9b^4-6a^9b-64a^9b^7-128a^9b^{10}\\&+64a^{12}b^4-4a^{12}b-256a^{12}b^7+16b^{10}-300b^6a^6-74b^3a^6)\\&(1+2ba)^2(1-2ba+4b^2a^2)^2=0.\end{aligned} \tag{27}$$

As $q\to 0$, the first factor vanishes faster than the second factor, whereas other factors does not vanish. This completes the proof. □

Theorem 3.2. *If $a := V(q)$ and $b := V(q^{10})$, then*

$$\begin{aligned}&64b^{12}a^6+32(5a^4-a)b^{11}+16(5a^2+20a^8-10a^5+32a^{11})b^{10}\\&+64(20a^9+10a^6-5a^3)b^9+20(34a^4-4a^7-a-48a^{10})b^8\\&+4(5a^2+160a^{11}-238a^5-20a^8)b^7+20(67a^6-4a^3+64a^9)b^6\\&+(10a^4-896a^{10}-760a^7)b^5+(80a^8+5a^2+120a^{11}+10a^5)b^4\\&+20(a^3-8a^6-11a^9)b^3+(112a^7+130a^{10}-15a^4)b^2\\&+(10a^5-a^2-20a^{11}-15a^8)b+a^{12}+b^6=0.\end{aligned} \tag{28}$$

Proof of the identity (28) is similar to the proof of the identity (21), except that in place of results (8) and (9), result (10) is used.

Theorem 3.3. *If $a := V(q)$ and $b := V(q^{14})$, then*

$$\begin{aligned}
&256b^{16}a^8 + 128(7a^4 - a)b^{15} + 256(32a^{15} + 28a^{12} + 7a^6)b^{14} + 1792(12a^8 - 3a^5 \\
&+ 16a^{11})b^{13} + 112(464a^{10} + 512a^{13} - a + 17a^4 - 8a^7)b^{12} + 448(32a^{15} - 3a^3 \\
&- 100a^{12} - 112a^9 + 20a^6)b^{11} + 112(512a^8 - 40a^5 - 256a^{14} + a^2 + 272a^{11})b^{10} \\
&+ 16(4480a^{13} - 1240a^7 + 3192a^{10} + 7a^4)b^9 + 112(56a^{15} - 453a^{12} + 128a^6 \\
&- 6a^3 - 259a^9)b^8 + 8(2513a^8 - 3200a^{14} - 336a^{11} - 392a^5)b^7 + 28(200a^{10} \\
&- 228a^7 + 432a^{13} + 29a^4)b^6 + 56(2a^3 - 17a^6 - 12a^9 + 4a^{12})b^5 + 7(a^2 - 16a^{11} \\
&- 453a^8 + 50a^5 + 464a^{14})b^4 + 56(20a^7 - 27a^{10} - 50a^{13} - 2a^4)b^3 + 4(203a^{12} \\
&+ 200a^9 - 28a^6 - 28a^{15})b^2 + (56a^{14} - a^2 - 49a^8 + 14a^5)b + a^{16} + b^8 = 0.
\end{aligned} \tag{29}$$

Proof of the identity (29) is similar to the proof of the identity (21), except that in place of results (8) and (9), results (11) and (12) are used.

Theorem 3.4. *If $a := V(q)$ and $b := V(q^{22})$, then*

$$\begin{aligned}
&b^{12} + a^{24} + 8(11b + 110b^4 + 225280b^{13} + 675840b^{16} + 1408b^{10} + 720896b^{19} \\
&+ 262144b^{22} - 6688b^7)a^{23} + 4(5046272b^{20} + 5406720b^{17} - 29744b^5 - 4302898b^{14} \\
&- 3906048b^{11} + 726704b^8 + 869b^2)a^{22} + 88(84800b^9 + 106496b^{18} + 425600b^{12} \\
&+ 525b^3 - 22812b^6 + 686080b^{15} - 262144b^{21})a^{21} + 22(131072b^{22} - 1080896b^{10} \\
&- 7456768b^{13} + 10606b^4 + 176464b^7 - 5b + 1277952b^{19} - 5062656b^{16})a^{20} \\
&+ 88(1449984b^{20} - 25502b^5 + 169b^2 + 2756608b^{17} + 913056b^{11} + 39612b^8 \\
&+ 2154880b^{14})a^{19} + 44(5703b^3 - 212080b^9 - 3180544b^{18} - 4571648b^{15} \\
&- 1245184b^{21} - 2875232b^{12} + 35626b^6)a^{18} + 44(614656b^{16} + 16384b^{22} - 20204b^7 \\
&- 19b + 1863680b^{19} + 905680b^{10} - 325632b^{13} - 11029b^4)a^{17} + 11(2205888b^{11} \\
&+ 23824384b^{17} + 102973344b^{14} + 161914880b^{20} + 4129b^2 - 39612b^5 - 435440b^8)a^{16} \\
&+ 88(1325b^3 - 1825024b^{18} - 1195552b^{12} + 13255b^6 - 2708288b^{15} - 319488b^{21} \\
&+ 4272b^9)a^{15} + 22(6697408b^{13} - 16889b^4 + 6679680b^{16} - 4096b^{22} + 1948416b^{10} \\
&- b + 2046976b^{19} - 226420b^7)a^{14} + 4(7879b^2 - 17135712b^{14} + 15851264b^{17} \\
&- 16050496b^{11} - 758274b^8 + 313863b^5 + 18625024b^{20})a^{13} + 2(7904512b^{18} \\
&- 36575b^3 + 6575536b^9 + 21602944b^{15} - 988361b^6 + 26518184b^{12} + 3942400b^{21} \\
&+ 2048b^{24})a^{12} + 8(5632b^{22} - 2302234b^{10} - 27984b^7 + 451616b^{16} + 40051b^4 \\
&+ 55b - 574016b^{13} - 1515008b^{19})a^{11} + 22(80448b^8 - 8192b^{20} + 670976b^{17} \\
&+ 389448b^{11} - 191b^2 - 16835b^5 - 2048b^{23} + 1058624b^{14})a^{10} + 44(8929b^6 + 335b^3 \\
&- 84634b^9 + 530432b^{18} + 391840b^{15} + 79360b^{21} - 122744b^{12})a^9 + 11(205824b^{19} \\
&- 4802b^7 - 2472b^4 - 15b - 41056b^{13} + 208740b^{10} + 3136b^{16} + 17408b^{22})a^8
\end{aligned}$$

$$+ 44(1346b^5 - 11633b^8 - 165056b^{17} + 22516b^{11} - 15b^2 - 80768b^{20} - 41936b^{14} - 1024b^{23})a^7 + 22(11228b^{12} - 157248b^{18} - 35328b^{21} - 132608b^{15} - 1553b^6 - 13b^3 + 14258b^9)a^6 + 22(910b^7 - 8608b^{13} - 12864b^{16} - 39b^4 - 3998b^{10} + b + 3072b^{22} - 8064b^{19})a^5 + 11(7b^2 - 354b^5 + 2048b^{23} + 3953b^8 + 26112b^{20} + 40384b^{17} - 13228b^{11} - 256b^{14})a^4 + 88(1104b^{18} - 78b^9 + 19b^6 - 175b^{12} + 384b^{21} + 620b^{15} - b^3)a^3 + 11(b^4 - 2b^{10} - 2b^7 - 1024b^{22} - 8b^{13} + 272b^{16} - 768b^{19})a^2 - ba^2 + 8(-256b^{23} - 88b^{17} - 352b^{20} + 11b^{14})a = 0. \tag{30}$$

Proof of the identity (30) is similar to the proof of the identity (21), except that in place of results (8) and (9), results (13) and (14) are used.

Theorem 3.5. *If $a := V(q)$ and $b := V(q^4)$, then*

$$4ab^3 + b - a^4 - 8a^3b^4 - 4a^3b - 6a^2b^2 = 0. \tag{31}$$

Proof. Employing the equation (17), we arrive at the equation (31). □

Theorem 3.6. *If $a := V(q)$ and $b := V(q^8)$, then*

$$a^8 - b - 28a^2b^3 - 64a^3b^7 - 16ab^5 + 8a^3b - 8ab^2 + 70a^4b^2 + 224a^5b^6 + 168a^5b^3 + 256a^6b^7 + 280a^6b^4 - 4a^6b + 128a^7b^8 + 128a^7b^5 + 8a^7b^2 = 0. \tag{32}$$

Proof. Employing the equations (17) and (31), we arrive at the equation (32). □

Theorem 3.7. *If $a := V(q)$ and $b := V(q^{12})$, then*

$$512b^{12}a^9 + 256(4a^9 + 6a^6 - a^3)b^{11} + 128(10a^9 - a^3 + 15a^6)b^{10} + 64(5a^3 - 6a^6 - 1 + 20a^9)b^9 + 32(8a^3 - 1 + 38a^9 + 8a^{12} + 57a^6)b^8 + 32(16a^{12} + 87a^6 - 2a^3 + 58a^9 - 2)b^7 + 8(39a^6 - 7 + 62a^3 + 218a^9 + 80a^{12})b^6 + 16(21a^6 - 4 + 26a^3 + 32a^{12} + 14a^9)b^5 + 2(332a^9 - 19 + 92a^3 + 152a^{12} + 498a^6)b^4 + (68a^3 - 19 + 128a^{12} + 868a^9 + 534a^6)b^3 + (40a^{12} - 5 + 210a^6 + 12a^3 + 140a^9)b^2 + (12a^3 - 20a^9 - 1 + 8a^{12} - 30a^6)b + a^{12} = 0. \tag{33}$$

Proof. Employing the equations (17) and (21), we arrive at the equation (33). □

Theorem 3.8. *If $a := V(q)$ and $b := V(q^{16})$, then*

$$32768a^{15}b^{16} + 16384(8a^{10} - 4a^{13} - a^7)b^{15} + 8192(71a^{11} + 4a^{14} - 8a^8)b^{14} + 4096(184a^{12} - a^3 - 64a^9 + 16a^{15} + 12a^6)b^{13} + 2048(199a^{13} + 88a^7 - 8a^4 - 160a^{10})b^{12} + 7168(24a^{14} - 5a^5 + 22a^{11} + 32a^8)b^{11} + 512(1032a^{12} + 59a^9$$

$$+ 88a^6 + 72a^{15} - 12a^3)b^{10} + 256(836a^7 - 1096a^{10} - a + 1556a^{13} - 88a^4)b^9$$
$$+ 128(931a^{14} - 2576a^{11} - 224a^5 - 8a^2 + 2712a^8)b^8 + 64(48a^{15} - 3052a^{12}$$
$$- 59a^6 - 64a^3 + 4770a^9)b^7 + 32(1092a^7 - 2166a^{13} + 6089a^{10} - 160a^4 - 8a)b^6$$
$$+ 16(5768a^{11} - 71a^2 + 154a^5 - 1224a^{14} + 2576a^8)b^5 + 8(3987a^{12} - 184a^3$$
$$+ 1032a^6 - 124a^{15} + 3052a^9)b^4 + 4(2166a^{10} - 199a^4 + 1556a^7 + 1220a^{13}$$
$$- 4a)b^3 + (2448a^{11} + 632a^{14} + 1862a^8 + 8a^2 - 336a^5)b^2 + (16a^{15} + 48a^9$$
$$- 1 + 16a^3 - 72a^6 + 124a^{12})b + a^{16} = 0. \tag{34}$$

Proof. Employing the equations (17) and (32), we arrive at the equation (34). □

ACKNOWLEDGEMENTS. The authors would like to thank University Grants Commission, New Delhi for their support under the Major Research Project No.F.No.34–140\2008 (SR).

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Received: November, 2011