Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 18, 849 - 856

Application of Fractional Calculus in Statistics

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Abstract: In this paper we present a new approach leads to innovative methods for estimating the parameters of distributions defined on R^+ by applying Fourier transform on Riemann-Liouville fractional differential and integral operator and by Mellin transform of the characterization function. These methods are useful specially when the density of the random variable has power-law tails. Estimating the class of complex moments using characteristic function, which include both integer and fractional moments, we show that random variable with power law distribution can be represented within this approach, even if its integer moments diverge.

Mathematics Subject Classification: 30C45

Keywords: Fourier Transform, Riemann-Liouville fractional differential and integral operator, Marchaud fractional derivative, Mellin transform, power-law distribution, first characterization of first kind, first characterization of second kind

Introduction

When the probability of measuring a particular value of some quantity varies inversely as a power of that value, the quantity is said to follow a power law. A power law distribution is a special kind of probability distribution. Fractional moments are very useful in dealing with random variable with power law distributions, $F(x) \sim |x|^{-\alpha}$, $\alpha > 0$ where $F(x)$ is the distribution function. In such cases, moments $E(|x|^p)$ exist only if $p < \alpha$ and integer order moments greater then α diverge. This type of problem arises in the distributions where power law statistics appear in many fields of applied science. For example, we mention the travel length distribution in human motion patterns and animal search processes [1]fluctuation in plasma devices [2], distribution of time scales in process such as the motion of charge carriers in amorphous semiconductors [8], scaling laws in polymer physics and its applications to gene regulation models [5], tracer dispersion in groundwater [7] and sticking times in turbulent flows [9]. For instance, the distributions of the sizes of cities, earthquakes, forest fires, solar flares, moon craters and people's personal fortunes all appear to follow power laws.

A continuous real variable with a power-law distribution has a probability $p(x)dx$ of taking a value in the interval from x to $x + dx$ given by,

$$p(x) = \frac{(\alpha - 1)}{x_{\min}} \left(\frac{x}{x_{\min}} \right)^{-\alpha}, \alpha > 1$$

In addition to cropping up as descriptions of many interesting quantities in social, biological and technological systems, power-law distributions have many interesting mathematical properties. Many of these come from the extreme right skewness of the distributions and the fact that only the first $[\alpha - 1]$ moments of a power-law distribution exist, all the rest are infinite. In general, the k^{th} moment is defined as

$$\begin{aligned} E(x^k) &= \int_{x_{\min}}^{\infty} x^k p(x) dx \\ &= {(\alpha - 1)}/{x_{\min}^{\alpha-1}} \int_{x_{\min}}^{\infty} x^{-\alpha+k} dx \\ &= x_{\min}^k \left(\frac{\alpha - 1}{\alpha - 1 - k} \right) \qquad \text{for } \ \alpha > k + 1 \end{aligned}$$

Thus, when $1 < \alpha < 2$, the first moment (the mean or average) is infinite, along with all the higher moments, when $2 < \alpha < 3$, the first moment is finite, but the second (the variance) and higher moments are infinite! In contrast, all the moments of the vast majority of other pdfs are finite. The lack of moments is of course a great limitation for the characterization of such type of distributions, because many methods of statistical analysis fails, like the characteristic function or the log characteristic function by moments or by cumulants, respectively.

In [4] the moments problems originally stated in terms of integer moments, has been extended to fractional moments where it has been shown that the knowledge of some fractional moments $E(X^q)$ improve significantly the convergence speed of the maximum entropy method for non negative densities. This can be explained by the non local of fractional moments.

Mellin transform naturally coincide with moments of the type $E(X^{\alpha-1})$ if the random variable X has a density defined in the positive domain. In this paper applying the Mellin transform to the *characterization function*, and not to the density, a sound representation of the statistics of random variable is possible. Here also by using fourier transform, simple properties of Riemann-Liouville fractional integral and derivative, we get a link between moments and the derivatives of the characteristic function evaluated in zero. Such a link is useful because it allows one to find existence criteria for fractional moments that can be derived in a simpler way with respect to [10] and clarify the non local nature of fractional moments.

Probabilistic characterization of random variables

Traditionally higher order moments of a random variable are generated from higher order derivatives of its characteristic function defined below. Let $F(x) = \Pr(X < x)$ is the cumulative distribution and $p(x) = \frac{dF(x)}{dx}$ is the probability density function (PDF). The Fourier transform of $p(x)$, denoted as $\mathcal{F}p = \varphi(u)$ is the *first characteristic function of first kind*, that is

$$\varphi(u) = \mathrm{E}\left(e^{iux}\right) = \int_{-\infty}^{\infty} e^{iux} p(x)dx \tag{1}$$

Where, $u \in \mathbb{R}$ and Expectation of the function $g(X) = X^k$with $k = 1, 2, 3\ldots$ provided they exist, give the integer moments of X. These integer moments indicated by $E(X^k)$ are related to the characteristic function by the Taylor expansion

$$\varphi(u) = \sum_{k=0}^{\infty} E\left\{(iX)^k\right\} \frac{u^k}{k!} \tag{2}$$

Due to the property

$$E\left\{(iX)^k\right\} = \left.\frac{d^k\varphi(u)}{du^k}\right|_{u=0} \tag{3}$$

Equation (2) is not always feasible due to divergent behavior of characteristic function as $u \to \infty$. The fundamentals of the generalization of above equations are briefly summarizd hereinafter and it will be shown that such representation does not entail the drawback of above equation no (2).
By taking account of properties of the Fourier transform, it is easy to show that moments of order v are obtained by derivation of the characteristic function:

$$m_\nu = \left.{}^{d^\nu\varphi(u)}\!/\!{}_{du^\nu}\right|_{u=0} \tag{4}$$

Which is same as (3) for integer values.
Now for complex moments, here we use fractional calculus method. Let $\nu = \beta + i\eta$, with $\beta > 0$ and $\eta \in \mathbb{R}$

$$\left(I_{\pm}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_{0}^{\infty} t^{\nu-1} f(x \mp t) dt \tag{5.1a}$$

$$\left(D_{\pm}^{\nu} f\right)(x) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_{0}^{\infty} t^{-\nu} f(x \mp t) dt \tag{5.1b}$$

$$\left(\mathbf{D}_{\pm}^{\nu} f\right)(x) = \frac{1}{\Gamma(-\nu)} \int_{0}^{\infty} t^{-\nu-1} \{f(x \mp t) - f(x)\} dt \tag{5.1c}$$

Where $(I_{\pm}^{\nu} f)$ and $(D_{\pm}^{\nu} f)$ denote the Riemann-Liouville (RL) fractional integral derivative, while $(\mathbf{D}_{\pm}^{\nu} f)$ is the Marchaud fractional derivative.
Fourier transform of fractional derivatives, in case of $0 < \beta < 1$, ([6], P.137) gives

$$\mathcal{F}\left\{\left(I_{\pm}^{\nu} f\right)(x); u\right\} = (\mp iu)^{-\nu} \left(\mathcal{F} f\right)(u) \tag{5.2a}$$

$$\mathcal{F}\left\{\left(D_{\pm}^{\nu} f\right)(x); u\right\} = (\mp iu)^{\nu} \left(\mathcal{F} f\right)(u) \tag{5.2b}$$

While inverse Fourier transform gives

$$\mathcal{F}^{-1}\left\{\left(I_{\pm}^{\nu} f\right)(x); u\right\} = (\pm iu)^{-\nu} \left(\mathcal{F}^{-1} f\right)(u) \tag{5.3a}$$

$$\mathcal{F}^{-1}\left\{\left(D_{\pm}^{\nu} f\right)(x); u\right\} = (\pm iu)^{-\nu} \left(\mathcal{F}^{-1} f\right)(u) \tag{5.3b}$$

Now we show that complex moments arise in a natural way once the fractional integral of the characteristic function is calculated in zero.

$$\mathcal{F}^{-1}\left\{\left(I_{\pm}^{\nu} \varphi\right)(u); x\right\} = (\pm ix)^{-\nu} \mathcal{F}^{-1}\left\{\varphi(u); x\right\} \tag{5.4a}$$

$$\mathcal{F}^{-1}\left\{\left(D_{\pm}^{\nu} \varphi\right)(u); x\right\} = (\pm ix)^{-\nu} \mathcal{F}^{-1}\left\{\varphi(u); x\right\} \tag{5.4b}$$

Choosing as Fourier pair $p(x)$ and $\varphi(u)$, defined in equ.(1) and letting $u = 0$ it is easy to obtain the generalized form of equ (3).

$$\left\{\left(I_{\pm}^{\nu} \varphi\right)(0); x\right\} = \int_{-\infty}^{\infty} (\pm ix)^{-\nu} p(x) dx = E\left\{(\pm iX)^{-\nu}\right\} \tag{5.5a}$$

$$\left\{\left(D_{\pm}^{\nu} \varphi\right)(0); x\right\} = \left\{(\mathbf{D}_{\pm}^{\nu} \varphi)(0); x\right\} = \int_{-\infty}^{\infty} (\pm ix)^{\nu} p(x) dx$$

$$= E\left\{(\pm iX)^{\nu}\right\} \tag{5.5b}$$

The set of complex moments is a natural generalization of integer moments as like as Riemannian- Liouville fractional differential operators generalize the classical differential calculus and (5.5b) coincide with equ.(3) when v assumes integer values. These equations relate the behavior of the characteristic function and the existence of moments in a more direct way than those proposed up to now in earlier literature (10). Moreover, in contrast to the local nature of the derivative and consequently of the integer moments, the complex moments are non local as like the fractional derivatives.
In [3] various generalization of the Taylor expansion presented in literature have been analyzed, showing the integral Taylor form based on inverse Mellin transform[6]. The initial point to find such a generalized Taylor form is to interpret the fractional derivative and integral defined in equs.(5.1) as a Mellin transform.

Mellin Transform Properties

Properties of Mellin Transform provide the formalism to derive higher order moments in a compact form. Let X be a positive-valued random variable whose pdf, $p(x)$, is defined over R^{+}. The *first characteristic function of the second kind* is expressed as the Mellin transform $\mathfrak{M}$ of $p(x)$

$$\varphi(s) = \mathfrak{M}[p(x); s] = \int_{0}^{\infty} x^{s-1} p(x) dx \tag{6}$$

provided that this integral converges, here $s = a + ib \in \mathbb{C}$ is the complex Laplace transform variable. We start from Mellin Transform of the characteristic function and get a link with equ.(5.1). Letting $x = 0$ equ. (5.1) can be written as

$$\Gamma(\nu)(I_{\pm}^{\nu}\varphi)(0) = \mathfrak{M}[\varphi(\mp t), \nu] = \int_{0}^{\infty} t^{\nu-1}\varphi(\mp t) dt \tag{7.1a}$$

$$\Gamma(-\nu)(\mathbf{D}_{\pm}^{\nu}\varphi)(0) = \mathfrak{M}[\{\varphi(\mp t) - \varphi(0)\}, -\nu] \tag{7.1b}$$

Where $\mathfrak{M}[., \nu]$ is the Mellin transform. As the Mellin transform has an inverse, knowing $\varphi(s)$, one can deduce $p(x)$, recalling the condition $\varphi(0) = 1$, we get two representation of the characteristic function given by

$$\varphi(\mp u) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(\nu)\left(I_{\pm}^{\nu}\varphi\right)(0) u^{-\nu} d\nu \tag{7.2a}$$

$$\varphi(\mp u) = 1 + \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(-\nu) \left(\mathbf{D}^{\nu}_{\pm}\varphi\right)(0) u^{\nu} d\nu \tag{7.2b}$$

Hold true, with $u > 0$ and where integral are performed along the imaginary axis with fixed real β that belong to the so called fundamental strip of the Mellin transform of the function $\varphi(u)$.
Such a representation gains an appealing touch once equ.(5.5) are considered, leading to

$$\varphi(\mp u) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(\nu) E\{(\pm iX)^{-\nu}\} u^{-\nu} d\nu \tag{7.3a}$$

$$\varphi(\mp u) = 1 + \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(-\nu) E\{(\pm iX)^{\nu}\} u^{\nu} d\nu \tag{7.3b}$$

With $u > 0$, the latter equations are the integral extensions of equ.(2) searched. Here presence of Gamma function in the inverse Mellin transform turns into a very amenable numerical treatment. The quite simple way we derived the latter relations entails many aspects about analytic functions, due to the complex nature of the inverse Mellin transform involved.A few remarks are therefore in order

1. Passing from eq.(7.3a) to eq.(7.3b) it does not suffice a change of sign because of the presence of the integrand residue in $v = 0$.

2. It is not difficult to show that the integrand of eq.(7.3a) might have, at most, poles for $\phi\,(-k + i0)\,, k = 0, 1, 2 \ldots$ and, conversely, the integrand in eq.(7.3b) might have, at most, poles for $\phi\,(k + i0)\,, k = 1, 2 \ldots$

3. The integral in eqs.(7.3) coincides with the sum of all the residues, and such sum can be proved to correspond to the r.h.s. of eq.(2).

Distributions with divergent moments can therefore be represented as sum of the residues of the integrand eqs.(7.3). Such an interesting theoretical aspect cannot be derived from classical applications of the inverse Mellin transform in probability, because in such works the starting point is the density and not the characteristic function.
Of course, the fundamental strip depends on the integrability of the CF, following eqs(7.2). It has been proved in [6] that the strictest fundamental strip associated with every absolute convergent CF, thus including α-stable random variables and other power-law distributions, is the interval $0 < \beta < 1$ if one uses eq.(7.3a). Then, in the following, we always assume such a restriction on

β.

The density function can be restored from eq.(7.3a) by inverse Fourier transform,

$$p(x) = \mathcal{F}^{-1}\{\varphi(u);x\} = \frac{1}{(2\pi)^2 i}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(\nu)\left(I_{+}^{\nu}\varphi\right)(0)\int_{0}^{\infty}u^{-\nu}e^{-iux}dud\nu$$

$$+\frac{1}{(2\pi)^2 i}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(\nu)\left(I_{-}^{\nu}\varphi\right)(0)\int_{-\infty}^{0}(-u)^{-\nu}e^{-iux}dud\nu \tag{8}$$

$$= \frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(\nu)\Gamma(1-\nu)E\left((-iX)^{-\nu}\right)(ix)^{\nu-1}d\nu$$

$$+\frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(\nu)\Gamma(1-\nu)E\left((iX)^{-\nu}\right)(-ix)^{\nu-1}d\nu \tag{9}$$

Eq.(9) represent the density function by the integral form of the Taylor approximation. Here the density is expressed in integral form without the need of partitioning, and no restriction on the non negativity of the density is needed. This fact is the consequence of having used the Mellin transform of the characterization function, since $\varphi(u) = \overline{\varphi(-u)}$where the overbar means conjugation. Evaluation of the CF for $u = 0$ suffices to restore the PDF in the form

$$p(x) = \frac{2}{(2\pi)^2}\mathrm{Re}\left\{\int_{\beta-i\infty}^{\beta+i\infty}\frac{\pi}{\sin(\pi\nu)}E\left\{(-iX)^{-\nu}\right\}(ix)^{\nu-1}d\nu\right\} \tag{10}$$

From eq(10) we can say that the PDF in terms of complex moments remains meaningful and computationally useful. Moreover, since the integration is performed only on the imaginary axis, i.e. the real part of v remains fixed in the integrand, $E\left\{(-iX)^{-\nu}\right\}$ exists also for distribution for which $E(X)^k$ does not exist, for k integer and greater than a certain value. By comparing eqs. (7.3) and (10) a perfect duality between the representation of the PDF and the CF is evidenced. The integrals in eqs(7.3a) and(10) remain independent of the choice of β within the strip $0 < \beta < 1$.

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Received: October, 2011