Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 15, 743 - 750

A Study on Iterative Solution of Population Dynamics Models

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Abstract

In this paper, simple approximate solutions of differential equations pertaining to population dynamics are obtained using iterative method. When the order of approximation tends to infinity, the solutions obtained converge towards an exact solution in some finite time interval.

Mathematics Subject Classifications: 34A34, 34B15, 34D10, 37M99

Keywords: Population dynamics, Iterative method, Lotka – Volterra, food web model, Taylor series

1. Introduction

When dealing with nonlinear differential equations, it is often the case that a closed form analytic solution for the differential equations of interest is normally unobtainable. Finding accurate and efficient methods for solving non linear problems has long been an active research undertaking. In this paper, an iterative technique is applied to obtain approximate solutions of population dynamics models. Here, we consider prey-predator interaction between two species and three species food web models for proving our cause.

Model 1: Prey-Predator interaction

We consider the system of equations of the form

$$\frac{dx_1}{dt} = x_1(a_1 - b_1x_1 - c_1x_2) \qquad (1.1)$$

$$\frac{dx_2}{dt} = x_2(a_2 - b_2x_1 - c_2x_2) \qquad (1.2)$$

where a_i, b_i, c_i (i = 1, 2) are positive constants. x_1 and x_2 represent the prey and predator population at time t.

Model 2: Food Web Model

Let x_1, x_2, x_3 be the densities of the three species forming a food web in a closed environment. Then, their evolution is governed by the set of equations

$$x_1' = x_1 (a_1 - b_1 x_1 - c_1 x_2) \qquad (1.3)$$

$$x_2' = x_2 (-a_2 + b_2 x_1 - c_2 x_2 - d_2 x_3) \qquad (1.4)$$

$$x_3' = x_3 (-a_3 + b_3 x_2 - c_3 x_3) \qquad (1.5)$$

where $a_i, b_i, c_i (i = 1, 2, 3)$ and d_2 are positive constants. The main feature of this paper is the method applied here is simple, approximate, tending towards exact solution of the above models.

2. Iterative solutions of first model

Transform the model equations (1.1) & (1.2) as

$$\frac{d}{dt}(\log x_1) = a_1 - b_1x_1 - c_1x_2 \qquad (2.1)$$

$$\frac{d}{dt}(\log x_2) = a_2 - b_2x_1 - c_2x_2 \qquad (2.2)$$

Changing variables as

$$u = \log x_1 \quad \Leftrightarrow \quad x_1 = \exp u \qquad (2.3)$$

$$v = \log x_2 \quad \Leftrightarrow \quad x_2 = \exp v \qquad (2.4)$$

u ,v represent real variables and are exponents of x, y respectively. Now,

$$\frac{du}{dt} = a_1 - b_1 \exp(u) - c_1 \exp(\nu) \tag{2.5}$$

$$\frac{d\nu}{dt} = a_2 - b_2 \exp(u) - c_2 \exp(\nu) \tag{2.6}$$

Let us suppose that the initial values are

$u(0) = u_0$ (2.7)

$v(0) = v_0$ (2.8)

Let the corresponding initial values be $x_1(0) = x_{10}$, $x_2(0) = x_{20}$.
Solutions of the system (2.5), (2.6) for conditions (2.7), (2.8) will be supposed in the following approximate form characteristic for nth approximation.

$u_n = A_0 + A_1 t + A_2 t^2 + .. + A_n t^n$ for n=0, 1, 2… (2.9)

$v_n = B_0 + B_1 t + B_2 t^2 + .. + B_n t^n$ for n=0, 1, 2… (2.10)

where A_n, B_n for n=0, 1, 2 .. are unknown real coefficients. These coefficients will be determined by supposition.

$A_0 = u_0$ (2.11)

$B_0 = v_0$ (2.12)

And by approximate iteration method

$$\frac{d}{dt} u_n = a_1 - b_1 \exp(u_{n-1}) - c_1 \exp(\nu_{n-1}) \quad \text{for } n = 1,2 \ldots\ldots \tag{2.13}$$

$$\frac{d}{dt} \nu_n = a_2 - b_2 \exp(u_{n-1}) - c_2 \exp(\nu_{n-1}) \quad \text{for } n = 1,2 \ldots\ldots \tag{2.14}$$

In the first order of approximation i.e. for n=1, using (2.11), (2.12) in (2.13), (2.14) yields

$A_1 = a_1 - b_1 \exp(u_0) - c_1 \exp(\nu_0)$ (2.15)

$B_1 = a_2 - b_2 \exp(u_0) - c_2 \exp(\nu_0)$ (2.16)

A_1 and B_1 are completely determined from equations (2.15), (2.16). In the second order of approximation i.e. for n=2, using 2.11), (2.12), (2.15), (2.16) in (2.13), (2.14) yields

$A_1 + 2A_2 t = a_1 - b_1 \exp(u_0 + A_1 t) - c_1 \exp(\nu_0 + B_1 t)$ (2.17)

$B_1 + 2B_2 t = a_2 - b_2 \exp(u_0 + A_1 t) - c_2 \exp(\nu_0 + B_1 t)$ (2.18)

Using additional conditions

$$|B_1 t| \leq 1, \quad |A_1 t| \leq 1 \tag{2.19}$$

and approximating by using Taylors expansion, the right hand side of (2.17), (2.18) yields

$$A_1 + 2A_2 t = a_1 - b_1 \exp(u_0)(1+A_1 t) - c_1 \exp(v_0)(1+B_1 t) \tag{2.20}$$

$$B_1 + 2B_2 t = a_2 - b_2 \exp(u_0)(1+A_1 t) - c_2 \exp(v_0)(1+B_1 t) \tag{2.21}$$

Using values of A_1 and B_1 from (2.15), (2.16) yields

$$A_2 = \frac{-b_1 A_1 \exp(u_0) - c_1 B_1 \exp(v_o)}{2} \tag{2.22}$$

$$B_2 = \frac{-b_2 A_1 \exp(u_o) - c_2 B_1 \exp(v_o)}{2} \tag{2.23}$$

From (2.22) and (2.23), A_2 and B_2 are completely determined. Using (2.9), (2.10) in (2.13), (2.14), the (n+1)th order of approximation is obtained as

$$A_1 + 2A_2 t + \ldots + (n+1) A_{n+1} t^n = a_1 - b_1 \exp(u_0 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_1 \exp(v_0 + B_1 t + B_2 t^2 + \ldots + B_n t^n) \text{ for } n = 2, 3 \ldots \tag{2.24}$$

$$B_1 + 2B_2 t + \ldots + (n+1) B_{n+1} t^n = a_2 - b_2 \exp(u_0 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_2 \exp(v_0 + B_1 t + B_2 t^2 + \ldots + B_n t^n) \text{ for } n = 2,3 \ldots \tag{2.25}$$

Suppose that the following additional approximate conditions are satisfied.

$$1 \geq |A_1 t + A_2 t^2 + \ldots + A_n t^n| \qquad \text{for } n = 2, 3 \ldots\ldots\ldots \tag{2.26}$$

$$1 \geq |B_1 t + B_2 t^2 + \ldots + B_n t^n| \qquad \text{for } n = 2, 3 \ldots\ldots\ldots \tag{2.27}$$

$\Leftrightarrow$

$$1 \geq |A_1 t| \geq |A_2 t^2| \geq \ldots\ldots.. \geq |A_n t^n| \qquad \text{for } n = 2,3 \ldots\ldots\ldots \tag{2.28}$$

$$1 \geq |B_1 t| \geq |B_2 t^2| \geq \ldots\ldots.. \geq |B_n t^n| \qquad \text{for } n = 2,3 \ldots\ldots\ldots \tag{2.29}$$

Conditions (2.28), (2.29) ensure the convergence of (2.9), (2.10) respectively. According to (2.26) to (2.29), the right hand sides of (2.24), (2.25) can be approximated by Taylors expansion

$$A_1 + 2A_2 t + \ldots + n A_n t^n = a_1 - b_1 \exp(u_0)(1 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_1 \exp(v_0)(1 + B_1 t + B_1 t^2 + \ldots + B_n t^n) \text{ for } n=2, 3 \ldots \tag{2.30}$$

$$B_1 + 2B_2 t + \ldots + n = a_2 - b_2 \exp(u_0)\,[1 + A_1 t + A_2 t^2 + .. + A_n t^n - c_2 \exp(v_0)\,[1 + B_1 t + B_1 t^2 + \ldots + B_n t^n] \quad \text{For } n=2, 3 \ldots \tag{2.31}$$

Then it follows that

$$A_n = \frac{-1}{n}(b_1A_{n-1}\exp(u_0) + c_1B_{n-1}\exp(v_0)) \tag{2.32}$$

$$B_n = \frac{-1}{n}(b_2A_{n-1}\exp(u_0) + c_2B_{n-1}\exp(v_0)) \quad \text{for } n=2, 3\ldots.. \tag{2.33}$$

In this way, coefficients A_n and B_n are completely determined by (2.32) & (2.33) respectively.

3. Iterative solution of second model

The model Equations (1.3), (1.4), (1.5) can be transformed as

$$\frac{d}{dt}(\log x_1) = a_1 - b_1x_1 - c_1x_2 \tag{3.1}$$

$$\frac{d}{dt}(\log x_2) = -a_2 + b_2x_1 - c_2x_2 - d_2x_3 \tag{3.2}$$

$$\frac{d}{dt}(\log x_3) = -a_3 + b_3x_2 - c_3x_3 \tag{3.3}$$

Changing variables as

$u = \log x_1 \Leftrightarrow x_1 = \exp(u)$ (3.4)

$v = \log x_2 \Leftrightarrow x_2 = \exp(v)$ (3.5)

$w = \log x_3 \Leftrightarrow x_3 = \exp(w)$ (3.6)

Now (3.1), (3.2) & (3.3) are

$$\frac{du}{dt} = a_1 - b_1\exp(u) - c_1\exp(v) \tag{3.7}$$

$$\frac{dv}{dt} = -a_2 + b_2\exp(u) - c_2\exp(v) - d_2\exp(w) \tag{3.8}$$

$$\frac{dw}{dt} = -a_3 + b_3\exp(v) - c_3\exp(w) \tag{3.9}$$

It is supposed that the initial values

$u(0) = u_0$ (3.10)

$v(0) = v_0$ (3.11)

$w(0) = w_0$ (3.12)

and the corresponding initial values are $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $x_3(0) = x_{30}$

Solutions of the system (3.7) to (3.9) for conditions (3.10), (3.12) will be supposed in the following approximate form characteristic for nth approximation.

$$u_{(n)} = A_0 + A_1t + A_2t^2 + .. + A_nt^n \quad \text{for } n=0,1,2\ldots \tag{3.13}$$

$v_{(n)} = B_0+B_1t+B_2 t^2+ .. + B_n t^n$ for n=0,1,2… (3.14)

$w_{(n)} = D_o+D_1t+D_2 t^2+ .. + D_n t^n$ for n=0, 1,2… (3.15)

Setting
$A_0 = u_0$ (3.16)
$B_0 = v_0$
(3.17)
$D_0 = W_0$ (3.18)
where A_n, B_n, D_n are unknown real coefficients.

Using approximate iteration method.

$$\frac{du_n}{dt} = a_1 - b_1 \exp(u_{n-1}) - c_1 \exp(v_{n-1}) \text{ for n=1,2 …..} \quad (3.19)$$

$$\frac{dv_n}{dt} = -a_2 + b_2 \exp(u_{n-1}) - c_2 \exp(v_{n-1}) - d_2 \exp(w_{n-1}) \text{ for n=1,2 ….} \quad (3.20)$$

$$\frac{dw_n}{dt} = -a_3 + b_3 \exp(v_{n-1}) - c_3 \exp(w_{n-1}) \text{ for n=1,2 ….} \quad (3.21)$$

In the first order of approximation i.e. for n=1, using (3.16) to (3.18) in (3.19) to (3.21) yields

$A_1 = a_1 - b_1\exp(u_0) - c_1\exp(v_0)$ (3.22)

$B_1 = -a_2 + b_2\exp(u_0) - c_2\exp(v_0) - d_2\exp(w_o)$ (3.23)

$D_1 = a_3 - b_3\exp(v_0) - c_3\exp(w_0)$ (3.24)

A_1, B_1 and D_1 are completely determined from equations (3.22) to (3.24). In the second order of approximation i.e. for n=2, using (3.16) to (3.18) in (3.22) to (3.24) yields

$A_1 + 2A_2 t = a_1 - b_1 \exp(u_0 + A_1 t) - c_1\exp(v_0 + B_1 t)$ (3.25)

$B_1 + 2B_2 t = -a_2 + b_2 \exp(u_0 + A_1 t) - c_2\exp(v_0 + B_1 t) - d_2 \exp(w_0 + D_1 t)$ (3.26)

$D_1 + 2D_2 t = a_3 - b_3 \exp(v_0 + B_1 t) - c_3\exp(w_0 + D_1 t)$ (3.27)

Using additional conditions

$$|A_1 t| \le 1, \quad |B_1 t| \le 1, \quad |D_1 t| \le 1$$

and approximating using Taylors expansion, the right hand side of (3.25), (3.26), (3.27) yields

$A_1 + 2A_2 t = a_1 - b_1 \exp(u_0)(1+A_1 t) - c_1 \exp(v_0)(1+B_1 t)$ (3.28)

$B_1 + 2B_2 t = -a_2 + b_2 \exp(u_0)(1+A_1 t) - c_2 \exp(\nu_0)(1+B_1 t) - d_2 \exp(w_0)(1+D_1 t)$ (3.29)

$D_1 + 2D_2 t = a_3 - b_3 \exp(\nu_0)(1+B_1 t) - c_3 \exp(w_0)(1+D_1 t)$ (3.30)

Making use of equations (3.22) to (3.24),

$$A_2 = \frac{-1}{2}(b_1 A_1 \exp(u_0) + c_1 B_1 \exp(\nu_0)) \quad (3.31)$$

$$B_2 = \frac{1}{2}(b_2 A_1 \exp(u_0) - c_2 B_1 \exp(\nu_0) - d_2 D_1 \exp(w_0)) \quad (3.32)$$

$$D_2 = \frac{1}{2}(b_3 B_1 \exp(\nu_0) + c_3 D_1 \exp(w_0)) \quad (3.33)$$

From 3.31 to 3.33, A_2, B_2 and D_2 are completely determined. Using (3.13), (3.14), (3.15) in (3.19), (3.20), (3.21), the (n+1)th order of approximation is obtained as

$$A_1 + 2A_2 t + \ldots + (n+1) A_{n+1} t^n = a_1 - b_1 \exp(u_0 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_1 \exp(v_0 + B_1 t + B_1 t^2 + \ldots + B_n t^n) \quad (3.34)$$

$$B_1 + 2B_2 t + \ldots + (n+1) B_{n+1} t^n = -a_2 + b_2 \exp(u_0 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_2 \exp(\nu_0 + B_1 t + B_1 t^2 + \ldots + B_n t^n) - d_2 (w_0 + D_1 t + D_2 t^2 + \ldots + D_n t^n) \quad (3.35)$$

$$D_1 + 2D_2 t + \ldots + (n+1) D_{n+1} t^n = -a_3 + b_3 \exp(\nu_0 + B_1 t + B_2 t^2 + .. + B_n t^n) - c_3 \exp(w_0 + D_1 t + D_1 t^2 + \ldots + D_n t^n) \quad (3.36)$$

Suppose that the following additional approximate conditions are satisfied.

$1 \geq | A_1 t + A_2 t^2 + \ldots + A_n t^n$ for n = 2,3 ……. (3.37)
$1 \geq | B_1 t + B_2 t^2 + \ldots + B_n t^n |$ for n = 2,3 ……. (3.38)
$1 \geq | D_1 t + D_2 t^2 + \ldots + D_n t^n|$ for n = 2,3 ……. (3.39)
$1 \geq | A_1 t| \geq |A_2 t^2| \geq \ldots \geq | A_n t^n|$ for n = 2,3 ……. (3.40)
$1 \geq | B_1 t| \geq |B_2 t^2| \geq \ldots \geq | B_n t^n|$ for n = 2,3 ……. (3.41)
$1 \geq | D_1 t| \geq |D_2 t^2| \geq \ldots \geq |D_n t^n |$ for n = 2,3 ……. (3.42)

Conditions (3.40) to (3.42) ensure the convergence of (3.13) to (3.15) respectively. According to (3.37) to (342), right hand sides of (3.34) to (3.36) can be approximated by Taylor's expansion

$A_1 + 2A_2 t + \ldots + nA_n t^n = a_1 - b_1 \exp(u_1)(1 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_1 \exp(v_0)[1 + B_1 t + B_1 t^2 + \ldots + B_n t^n)$ n=2, 3 … (3.43)

$B_1 + 2B_2 t + \ldots + n B_n t^n = -a_2 + b_2 \exp(u_0)(1 + A_1 t + A_2 t^2 + .. + A_n t^n) - c_2 \exp(v_0)[1 + B_1 t + B_2 t^2 + \ldots + B_n t^n)] - d_2 \exp(w_0)(1 + D_1 t + D_2 t^2 + \ldots + D_n t^n)$ (3.44)

$D_1 + 2D_2t + \ldots + n\ D_n t^n = -a_3 + b_3 \exp(v_0)(1 + B_1 t + B_2 t^2 + .. + B_n t^n)$
$-c_3 \exp(w_0)[1 + D_1 t + D_2 t^2 + \ldots + D_n t^n)$ (3.45)

Then it follows that

$$A_n = \frac{-1}{n}(b_1 A_{n-1} \exp(u_0) + c_1 B_{n-1} \exp(v_0)) \quad \text{for } n=2,3\ldots \quad (3.46)$$

$$B_n = \frac{1}{n}[b_2 A_{n-1} \exp(u_0) - c_2 B_{n-1} \exp(v_0) - d_2 D_{n-1} \exp(w_0) \quad \text{for } n=2,3. \quad (3,47)$$

$$D_n = \frac{1}{n}(b_3 B_{n-1} \exp(v_0) - c_3 D_{n-1} \exp(w_0) \quad \text{for } n=2,3\ldots \quad (3.48)$$

In this way coefficients A_n, B_n & D_n and completely determined by (3.46) to (3.48) respectively. Approximate solutions (2.9), (2.10), (3.15) of the systems (2.5), (2.6), (3.19) to (3.21) for conditions (3.16) to (3.18) are consistent and convergent in a time interval [0, t] that is not infinitesimally small. Obviously when $n \to \infty$, approximate analytical solutions of the above systems with the given conditions tends towards exact solutions.

4. Conclusions

In this paper, the population dynamics model is studied and approximate solutions of differential equations pertaining to the model are obtained using iterative method. When the order of approximation tends to infinity, the solutions obtained converge towards an exact solution in some finite time interval which is not infinitesimally small.

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