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# Some Classes of Composition Operators on the Fock Space

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**Abstract**

In this paper normal, quasinormal, hyponormal, quasihyponormal, class A and paranormal composition operators on Fock space are characterized.

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**Keywords:** Fock space, composition operator, normal, quasinormal, hyponormal, quasi hyponormal , class A and paranormal operators.

## 1. Introduction

Composition operators on spaces of analytic functions have been studied in many settings. Much has been written about the properties of these operators on the Hardy, Bergman, and Bloch spaces on the unit disk in the complex plane or on the unit ball in $C^n$.( [2] ,[5] and [6] ). Already in the paper [1] bounded and compact composition operator on Fock space is discussed. We will determine which composition operators are normal , quasinormal ,hyponormal, quasi hyponormal, class A and paranormal.

The Fock space $F$ is the Hilbert space of all holomorphic functions on $C^n$ with inner product

$$\langle f,g\rangle = \frac{1}{(2\pi)^n}\int_{C^n} f(z)\overline{g(z)}e^{\frac{-1}{2}|z|^2}dv(z)$$

where $v$ denotes Lebesgue measure on $C^n$. (refer [1] and [4] ).

Let $e_n(z)=\sqrt{\frac{1}{n!}}z^n$ for a positive integer n. Then $\{e_n\}$ forms an orthonormal basis for $F$. Since each point evaluation is a bounded linear functional on $F$, for $w\in C^n$ there exists a unique function $k_w\in F$ such that $\langle f,k_w\rangle = f(w)$ which holds for all $f\in F$. The reproducing kernel functions for the Fock space are given by

$$k_w(z)=e^{\langle z,w\rangle/2}$$

where $\langle z,w\rangle=\sum_1^n z_j\overline{w_j}$ . Note that the substitution $f=k_w$ into the reproducing formula $\langle f,k_w\rangle = f(w)$ which holds for all $f\in F$ and $w\in C^n$ leads to the identity $\|k_w\|=\exp(|w|^2/4)$. Through out this paper we use $f=k_w$ for the reproducing kernel function for the Fock space $F$ and let $k_0$ = 1 be the point evaluation on $F$ ( [3] and [4] ).

For a given holomorphic mapping $\varphi : C^n\to C^n$ , the composition operator $C_\varphi : F\to F$ is given by $C_\varphi(f)=f\circ\varphi$. In the paper [4] it is proved that if the operator $C_\varphi$ is bounded ,then $\varphi$ must be of the form $\varphi(z)=Az+B$ where A is an $n\times n$ matrix and B is an $n\times 1$ vector . Further more it will follow that $\|A\| \leq 1$ for bounded $C_\varphi$ and that B will be restricted by the condition that $\langle A\varsigma, B\rangle = 0$ for any $\varsigma$ in $C^n$ with $|A\varsigma| = |\varsigma|$ . In paper [1] Theorem 1 shows that if $C_\varphi$ is compact , then $\|A\| < 1$ with no restriction on B . In section 3 we characterize normal and other classes of composition operators $C_\varphi$ where $C_\varphi$ is bounded .

## 2. PRILIMINARIES

$C_\varphi$ be a composition operator on the Fock space $F$ . Then $C_\varphi$ is normal iff $C_\varphi^* C_\varphi = C_\varphi C_\varphi^*$ In paper [1] it is shows that $C_\varphi^* = M_{k_B} C_\tau$ where $\tau(z) = A^* z$ and $M_{k_B}$ is multiplication by the kernel function $k_B$ .We say $C_\varphi$ is quasinormal iff $C_\phi (C_\varphi^* C_\phi) = (C_\phi^* C_\phi) C_\phi$ , hyponormal iff $C_\varphi^* C_\varphi \geq C_\varphi C_\varphi^*$ ,quasi hyponormal iff $(C_\varphi^*)^2 C_\varphi^2 \geq (C_\varphi^* C_\varphi)^2$ , ‘Class A’ operator iff $(C_\varphi^* |C_\varphi|^2 C_\varphi)^{1/2} \geq C_\varphi^* C_\varphi$ and paranormal iff $C_\phi^{*2} C_\phi^2 + 2kC_\phi^* C_\phi + k^2 I > 0$ , $k > 0$ .

## 3. MAIN RESULTS :

**Theorem 3.1**

Let $C_\varphi$ be a composition operator on $F$ .Then $C_\varphi$ is normal if and only if $M_{k_B} C_{\varphi o \tau} = M_{k_B o \phi} C_{\tau o \phi}$ .

**Proof :**

$C_\varphi$ is normal if and only if $C_\varphi^* C_\varphi = C_\varphi C_\varphi^*$ where $\phi(z) = Az + B$ so that $C_\varphi$ is bounded. Then $C_\varphi^* = M_{k_B} C_\tau$ where $\tau(z) = A^* z$ and $M_{k_B}$ is multiplication by the kernel function $k_B$ .

Since $C_\phi^* = M_{k_B} C_\tau$ [1] it follows that

$$M_{k_B} C_\tau C_\phi = C_\phi M_{k_B} C_\tau$$

Since $C_\varphi M_\theta = M_{\theta o \phi} C_\varphi$ we have

$$M_{k_B} C_{\varphi o \tau} = M_{k_B o \phi} C_{\tau o \phi}$$

as desired.

**Corollary 3.2**
Let $C_\varphi$ be a composition operator on $F$ .If $C_\varphi$ is normal then $M_{k_B} = M_{k_B o \phi}$ .

**Proof :** By the above theorem , $C_\varphi$ is normal if and only if

$$M_{k_B} C_{\varphi o \tau} = M_{k_B o \phi} C_{\tau o \phi}$$

$$M_{k_B} C_{\varphi o \tau}(f) = M_{k_B o \phi} C_{\tau o \phi}(f) \text{ for all } f \in F .$$

$$M_{k_B}(f o \phi o \tau) = M_{k_B o \phi}(f o \tau o \phi)$$

Let $f = k_0 = 1$ be the point evaluation on $F$ .

We have $M_{k_B}(k_0 o \phi o \tau) = M_{k_B o \phi}(k_0 o \tau o \phi)$

$$M_{k_B} = M_{k_B o \phi}$$

**Corollary 3.3**
Let $C_\varphi$ be a composition operator on $F$ where $\phi(z) = Az$ , $A$ is n × n matrix. Then $C_\varphi$ is normal if and only if $A^* A = AA^*$

**Proof :** $\phi o \tau(z) = \tau o \phi(z)$ is equivalent to $A^* Az = AA^* z$ . Since $k_B = 1$, the result follows.

**Theorem 3.4** Let $C_\varphi$ be a composition operator on $F$ .Then $C_\varphi$ is hyponormal if and only if $M_{k_B} C_{\varphi o \tau} \geq M_{k_B o \phi} C_{\tau o \phi}$ .

**Proof :**
$C_\varphi$ is hypo normal if and only if

$$C_\varphi^* C_\varphi \geq C_\phi C_\phi^*$$

Since $C_\phi^* = M_{k_B} C_\tau$ it follows that

$$M_{k_B} C_{\varphi o \tau} \geq M_{k_B o \phi} C_{\tau o \phi}$$

**Corollary 3.5**
$C_\varphi$ be a composition operator on $F$ and let $f = k_w$ be the reproducing kernel function for the Fock space and put f = $k_0 = 1$ be the point evaluation on $F$ .
If $C_\varphi$ is hyponormal then $M_{k_B} \geq M_{k_B o \phi}$ .

**Theorem 3.6** Let $C_\varphi$ be a composition operator on $C_\varphi$. Then $C_\varphi$ is quasi normal if and only if $(M_{k_B o\phi})C_{\phi o\tau o\phi} = M_{k_B} C_{\phi o\phi o\tau}$.

**Proof :** $C_\varphi$ is quasi normal if and only if

$$C_\phi (C_\varphi{}^* C_\phi) = (C_\phi{}^* C_\phi) C_\phi$$

But $C_\phi (C_\phi{}^* C_\phi) = C_\phi (M_{k_B} C_{\phi o\tau}) = M_{k_B o\phi} C_{\phi o\tau o\phi}$

and $(C_\phi{}^* C_\phi) C_\phi = C_\phi{}^* C_{\phi o\phi} = M_{k_B} C_\tau C_{\phi o\phi} = M_{k_B} C_{\phi o\phi o\tau}$

so $C_\varphi$ is quasi normal if and only if

$$(M_{k_B o\phi}) C_{\phi o\tau o\phi} = M_{k_B} C_{\phi o\phi o\tau}.$$

**Corollary 3.7**

Let $C_\varphi$ be a composition operator on $F$ where $\phi(z) = Az$ and $A$ is n × n matrix. Then $C_\varphi$ is quasinormal if and only if A commutes with $A^* A$.

**Proof :** By the above theorem , $C_\varphi$ is quasinormal if and only if

$C_{\phi o\tau o\varphi} = C_{\phi o\phi o\tau}$ which is equivalent to $AA^* A = AAA^*$

**Theorem 3.8** Let $C_\varphi$ be a composition operator on $F$ .Then $C_\varphi$ is quasihyponormal if and only if $M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}} \le M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$ where $\phi^{(2)} = \phi o \phi$.

**Proof :**

$C_\varphi$ is quasihyponormal if and only if

$$C_\phi{}^{*^2} C_\phi{}^2 \ge (C_\phi{}^* C_\phi)^2$$

Now $(C_\phi{}^* C_\phi)^2 = (M_{k_B} C_{\phi o\tau})^2$

$$= M_{k_B} C_{\phi o\tau}\, M_{k_B} C_{\phi o\tau}$$

$$= M_{k_B} M_{k_B o\phi o\tau} C_{\phi o\tau o\phi o\tau}$$
$$= M_{k_B} M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}}$$

and $C_\phi^{*^2} C_\phi^{\ 2} = C_\phi^{\ *}(M_{k_B} C_{\phi o\tau}) C_\phi$
$$= C_\phi^{\ *} M_{k_B} C_{\phi o\phi o\tau}$$
$$= M_{k_B} C_\tau M_{k_B} C_{\phi o\phi o\tau}$$
$$= M_{k_B} M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$$

and so the composition operator on $C_\varphi$ on $F$ is quasihyponormal if and only if
$$M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}} \le M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}} \quad .$$

**Corollary 3.9**

Let $C_\varphi$ on $F$ be a composition operator where $\phi(z) = Az + B$.Then $C_\varphi$ is quasihyponormal if and only if $(AA^*)^2 \le A^2\, A^{*^2}$ and $AB \ge 0$.

**Proof :**

Observe that $(\phi o\tau)^2(z) = (AA^*)^2 z + B$

$\phi^2 o\tau^2(z) = A^2 A^{*^2} z + AB + B$ and hence the result.

**Theorem 3.10** Let $C_\varphi$ be a composition operator on $F$. Then $C_\varphi$ belongs to class A operator if and only if $C_\varphi$ is quasihyponormal .

**Proof :**

$C_\varphi$ is of class A operator if and only if

$(C_\phi^{\ *} C_\phi)^2 \le C_\phi^{\ *} \left| C_\phi \right|^2 C_\phi$ and so

$(C_\phi^{\ *} C_\phi)^2 \le C_\phi^{\ *2} C_\phi^{\ 2}$

Now $(C_\phi^{\ *} C_\phi)^2 = M_{k_B} M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}}$

and $(C_\phi^{\ *2} C_\phi^{\ 2}) = M_{k_B} M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$

which reduces to $M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}} \le M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$

hence it is quasihyponormal.

**Theorem 3.11** $C_\varphi$ , a composition operator on $F$ is paranormal if and only if $C_\varphi$ is quasihyponormal.

**Proof** : $C_\varphi$ is paranormal if and only if $C_\phi^{*^2} C_\phi^{\ 2} + 2kC_\phi^{\ *} C_\phi + k^2 \geq 0$, for all real k which is equivalent to

$$M_{k_B} M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}} + 2kM_{k_B} C_{\phi o\tau} + k^2 \geq 0$$

Which shows that

$$(M_{k_B} C_{\phi o\tau})^2 \leq M_{k_B} M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$$

which reduces to

$$M_{k_B o\phi o\tau} C_{(\phi o\tau)^{(2)}} \leq M_{k_B o\tau} C_{\phi^{(2)} o\tau^{(2)}}$$

Using the previous theorem , the result follows.

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