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On I-Convergence of Double Sequences in 2-Normed Spaces

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Abstract

The concept of I-convergence was introduced by Kostyrko et al (2001).It seems therefore reasonable to investigate the concept of I -convergence for the double sequences in 2-normed spaces.In this article we define and investigate ideal analogue of convergence for double sequences in 2-normed space and so we extend this concepts to I_2-limit points and I_2-cluster points in this spaces.We prove some basic properties.

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1. Introduction

The notion of statistical convergence was introduced first by Fast[7].The idea of I-convergence was introduced by Kostyrko et al (2001) and also independently by Nuray and Ruckle (2000) , as a generalization of statistical convergence

(Fast (1951), Schoenberg(1959)).

Very recently some works on I-convergence of double sequences have also been done (see [5,6,21,27]) . The concept of linear 2-normed spaces has been investigated by Gâhler in1960's [11,12] and has been developed extensively in different subjects by others[3,14,23,25].Note that the notion of the statistical convergence for the double sequences in 2-normed spaces was introduced in papers [24] .It seems therefore reasonable to investigate the concept of I-convergence for the double sequences in 2-normed spaces.

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|.\|)$ be a normed space. Let E be subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$.The quotient $d_j(E) = card(E \bigcap \{1, ..., j\})/j$ is called the j'th *partial density* of K.Note that d_j is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, ..., j\}$[2,4,7,24].

The limit d(E)=$\lim_{j\to\infty} d_j(K)$ is called the *natural density* of $E \subseteq \mathbb{N}$(if exists). Clearly, finite subsets have natural density zero and $d(E^c) = 1 - d(E)$ where $E^c = E - \mathbb{N}$, i.e., the complement of E [2,27].

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be statistically convergent to $l \in X$if the set$A(\epsilon)$=$\{n \in \mathbb{N} : \|x_n - l\| \geq \epsilon\}$has natural density zero for each $\epsilon > 0$ in other words for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} card(\{k \leq n : |x_k - l| \geq \varepsilon\}) = 0$$

and x=$(x_n)_{n\in\mathbb{N}}$ is called to be statistically Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n} card(\{k \leq n : |x_k - x_N| \geq \varepsilon\}) = 0$$

The convergence of a double sequence introduce by many manner[4,21,22]. By the convergence of a double sequence we mean the convergence in Pringsheim's sense [22]. A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ is called to be *convergent in the Pringsheim's sense* if for each $\varepsilon > 0$ there exist a positive integer $N = N(\varepsilon)$ such that for all $j, k \geq N$ implies $|x_{jk} - l| < \varepsilon$. L is called the Pringsheim limit of x.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of positive integers and let $A(n, m)$ be the numbers of (j, k) in A such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of *natural density* can be defined as follows.

The *lower asymptotic density* of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{d_2}(A) = \liminf_{n,m} \frac{A(n, m)}{nm}$$

If the sequence $(\frac{A(n,m)}{nm})_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that A has a *double natural density* and is defined as

$$d_2(A) = \lim_{n,m} \frac{A(n,m)}{nm}$$

Next we recall the following definition ,where Y represents an arbitrary set.

Definition 1.1. A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets a nonempty set Y is said to be an ideal in Y if:
i) $\emptyset \in \mathcal{I}$
ii) $A, B \in \mathcal{I}$ implies $A \bigcup B \in \mathcal{I}$
iii) $A \in \mathcal{I}, B \subseteq A$ implies $B \in \mathcal{I}$
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Definition 1.2. Let $Y \neq \emptyset$.Anon empty family F of subsets of Y is said to be a *filter* in Y provided:
i) $\emptyset \in F$.
ii) $A, B \in F$ implies $A \cap B \in F$.
iii)$A \in F, A \subseteq B$ implies $B \in F$.
If $\mathcal{I}$ is a nontrivial ideal in Y,$Y \neq \emptyset$,then the class

$$F(\mathcal{I}) = \{M \subset Y : (\exists A \in \mathcal{I})(M = Y - A)\}$$

is a filter on Y ,called the *filter associated* with $\mathcal{I}$.

Definition 1.3. A nontrivial ideal $\mathcal{I}$ in Y is called *admissible* if $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Definition 1.4. A nontrivial ideal $\mathcal{I}$ in $\mathbb{N} \times \mathbb{N}$ is called *strongly admissible* if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$.
It is evident that a strongly admissible ideal is admissible also.
Let $\mathcal{I}_0$=$\{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \in \mathbb{N} \times \mathbb{N} - A)\}$.
Then $\mathcal{I}_0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}$ is strongly admissible if and only if $\mathcal{I}_0 \subseteq \mathcal{I}$.[5]

Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$.The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if
for each $\epsilon > 0$ the set $A(\epsilon)$=$\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to $\mathcal{I}$ [1,17,19].

2. Preliminary Notes

The concept of $\mathcal{I}$-convergence of double sequences in metric spaces X is defined as follow.

Definition 2.1. A double sequence $x=(x_{jk})_{j,k\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $l \in X$ if for every $\varepsilon > 0$ we have $A(\varepsilon) \in I$,
where $A(\varepsilon)=\{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, l) \geq \varepsilon\}$and we write it as

$$\mathcal{I} - \lim_{m,n} x_{mn} = l$$

The notion of linear 2-normed spaces has been investigated by Gâhler in1960's [11,12] and has been developed extensively in different subjects by others[3,14,23]. Let X be a real linear space of dimension greater than 1,and $\|.,.\|$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:

G1)$\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

G2) $\|x, y\| = \|y, x\|$ for all x, y in X.

G3)$\|\alpha x, y\| = |\alpha|\|x, y\|$ where α is real

G4)$\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X

$\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a linear 2-normed space. In addition, for all scalars α and all x, y, z in X, we have the following properties :

1)$\|.,.\|$ is nonnegative.

2)$\|x, y\| = \|x, y + \alpha x\|$

3)$\|x - y, y - z\| = \|x - y, x - z\|$

Some of the basic properties of 2-norm introduce in [23].
As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given clearly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1| \quad , \quad x = (x_1, x_2) \quad y = (y_1, y_2) \tag{2.1}$$

Given a 2-normed space $(X, \|.,.\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in X$ .This can be written by the formula:

$$(\forall z \in Y)(\forall \epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \qquad \|x_n - x, z\| < \epsilon$$

We write it as

$$x_n \xrightarrow{\|.,.\|_X} x$$

Definition 2.2. A sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a 2-normed space $(X, \|.,.\|)$ if $\lim_{n,m} \|x_n - x_m, z\| = 0$ for every $z \in X$.
Recall that $(X, \|.,.\|)$ is a 2-Banach space, if every Cauchy sequence in X is convergence to some $x \in X$.

Definition 2.3. Let $(X, \|.,.\|)$ be 2-normed space and $x \in X$.We say that x is an *accumulation point* of X if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of X such that $x_k \neq x$ (for any k) and $x_n \xrightarrow{\|.,.\|_X} x$

Lemma 2.4. *[15]Let $v = \{v_1, ..., v_k\}$ be a basis of X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty} \|x_n - x, v_i\| = 0$ for every $i = 1, ..., k$. We can define the norm $\|.\|_\infty$ on X by*

$$\|x\|_\infty := \max\{\|x, v_i\| : i = 1, ..., d = k\}$$

Associated to the derived norm $\|.\|_\infty$, we can define the(open) balls $B_{v_1,v_2,...,v_n}(\boldsymbol{x}, r)$=$B_v(\boldsymbol{x}, r)$ centered at $\boldsymbol{x}$ having radius r by

$$B_v(\boldsymbol{x}, r) := \{y : \|\boldsymbol{x} - y\|_\infty < r\}$$

Lemma 2.5. *[15]A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$*

Example 2.6. Let $X = \mathbb{R}^2$ be equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1| \qquad , \qquad x = (x_1, x_2) \qquad y = (y_1, y_2)$$

Take the standard basis $\{i, j\}$ for $\mathbb{R}^2$.

Then, $\|x, i\| = |x_2|$ and $\|x, j\| = |x_1|$, and so the derived norm $\|.\|_\infty$ with respect to $\{i, j\}$ is

$$\|x\|_\infty = \max\{|x_1|, |x_1|\}, \qquad x = (x_1, x_2)$$

Thus, here the derived norm $\|.\|_\infty$ is exactly the same as the uniform norm on $\mathbb{R}^2$.Since the derived norm is norm,it is equivalent to Euclidean norm on $\mathbb{R}^2$.

3. I_2-LIMIT POINTS AND I_2-CLUSTER POINTS IN 2-NORMED SPACES

In [13] , the concepts of an ordinary limit points and I-limit points for a single sequences was generalized in 2-normed spaces.
In this section ,we define I_2-convergence for double sequence in 2-normed spaces and so we extend this concepts to I_2-limit points and I_2-cluster points in this spaces.For the following definition we were inspired by Pringsheims [22]

Definition 3.1. Let x=$(x_{jk})_{j,k\in\mathbb{N}}$ be a double sequence in 2-normed space $(X, \|.,.\|)$.A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ is said to be *convergent to* $l \in X$ if

$(\forall z \in X)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall j,k \geq N) \qquad \|x_{jk} - l, z\| < \varepsilon$
We write it as

$$x_{jk} \xrightarrow{\|.,.\|_X} l$$

A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ is said to be bounded if for each nonzero $z \in X$ and for each $j,k \in \mathbb{N}$ there exists $M > 0$ such that $\|x_{jk}, z\| < M$. Note that a convergent double sequence need not be bounded.

Now we define the I_2 and I_2^*-convergence for double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ as follows:

Definition 3.2. A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be I_2-convergence to $l \in X$,if for all $\varepsilon > 0$ and nonzero $z \in X$,the set

$$A(\varepsilon) = \{(j,k) : \|x_{jk} - l, z\| \geq \varepsilon\} \in I_2$$

In this case we write it as

$$I_2 - \lim_{j,k} x_{jk} = l$$

Remark 3.3. Put I_d= $\{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\}$. Then I_d is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and I_{d2}-convergence becomes statistical convergence[24].

Remark 3.4. Not that if I is the ideal I_0 then I_2-convergence coincide with the usual convergence(Definition 3.1).

Remark 3.5. If x=$(x_{jk})_{j,k\in\mathbb{N}}$ is I_2-convergent,then $(x_{jk})_{j,k\in\mathbb{N}}$need not be convergent .Also it is not necessarily bounded.This actuality can be seen from the next example.

Example 3.6. Let $(X, \|.,.\|)$ be 2-normed space introduced in Example 2.6 and the x=$(x_{jk})_{j,k\in\mathbb{N}}$ be defined as

(1,1) otherwise

and let $\qquad l = (1,1).$

Then for every $\varepsilon > 0$ and $z \in X$
$\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \subseteq \{1, 4, 9, 16, ..., j^2, ...\} \times \{1, 4, 9, 16, ..., k^2, ...\}$.
Hence
the cardinality of the set $\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \leq \sqrt{j}\sqrt{k}$ for

each $\varepsilon > 0$.
This implies $d_2(\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\}) = 0$ for each $\varepsilon > 0$ and $z \in X$.We have

$$I_{d2} - \lim_{j,k} x_{jk} = l$$

.
But x=$(x_{jk})_{j,k\in\mathbb{N}}$ is neither convergent to l nor bounded.

Remark 3.7. The following corollary can be verifies that if x=$(x_{jk})_{j,k\in\mathbb{N}}$ be I_2-convergent to $l \in X$, then l is determined uniquely.

Corollary 3.8. *let* x=$(x_{jk})_{j,k\in\mathbb{N}}$ *is a convergent double sequence in 2-normed space* $(X, \|.,.\|)$ *and* $l_1, l_2 \in X$.*If* $I_2\text{-}\lim_{j,k} \|x_{jk} - l_1, z\| = 0$ *and*
$I_2\text{-}\lim_{j,k} \|x_{jk} - l_2, z\| = 0$ *then* $l_1 = l_2$.

proof:*Let* $l_1 \neq l_2$,*hence there exists* $z \in X$ *such that* $0 \neq l_1 - l_2$ *and* z *are linearly independent.Put*

$$\|l_1 - l_2, z\| = 2\varepsilon, with \varepsilon > 0$$

Now

$$2\varepsilon = \|l_1 - x_{jk} + x_{jk} - l_2, z\| \leq \|x_{jk} - l_1, z\| + \|x_{jk} - l_2, z\|$$

Therefor

$$\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \subseteq \{(j,k) : \|x_{jk} - l_1, z\| \geq \varepsilon\} \in I$$

Hence $\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \in I$ *that is contradict with nontrivial* I.

Corollary 3.9. *If* $(x_{jk})_{j,k\in\mathbb{N}}$,$(y_{jk})_{j,k\in\mathbb{N}}$ *be double sequences in 2-normed space* $(X, \|.,.\|)$ *and* $I_2\text{-}\lim_{j,k} x_{jk} = a$,$I_2\text{-}\lim_{j,k} y_{jk} = b$ *then*
(i) $I_2\text{-}\lim_{j,k} x_{jk} + y_{jk} = a + b$
(ii) $I_2\text{-}\lim_{j,k} \alpha x_{jk} = \alpha a$ *,where* $\alpha \in \mathbb{R}$
proof(i):*Let* $\varepsilon > 0$. *For each nonzero* $z \in X$ *we have*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(x_{mn}+y_{mn})-(a+b), z\| \geq \varepsilon\} \subseteq \Big(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn}-a, z\| \geq \frac{\varepsilon}{2}\}$$

$$\bigcup\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|y_{mn} - b, z\| \geq \tfrac{\varepsilon}{2}\}\Big) \in I$$

Hence $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(x_{mn}+y_{mn})-(a+b), z\| \geq \varepsilon\} \in I$ *and the statements is follows.*
(ii) *The statement is an easy consequence of* (i)

Definition 3.10. Let $K \subseteq \mathbb{N}\times\mathbb{N}$ such that for each $(j,k) \in \mathbb{N}\times\mathbb{N}$ there exists $(m,n) \in K$ such that $(m,n) > (j,k)$ with respect to the dictionary ordering.If $x = (x_{jk})_{j,k\in\mathbb{N}}$ is a double sequence in X, then we call $(x)_K = \{x_{mn} : (m,n) \in K\}$as a *subsequence* of $(x_{jk})_{j,k\in\mathbb{N}}$.

Definition 3.11. An element $l \in X$ is said to be *limit point* of a double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ if there exists a subsequence of x which is convergent to l.

Example 3.12. Let $(X, \|.,.\|)$ be 2-normed space introduced in Example 3.6 and the x=$(x_{jk})_{j,k\in\mathbb{N}}$ be defined as

(1,k) otherwise

and let $l = (1,1)$.

We put $K = \{(m,m) : m \in \mathbb{N}\}$. Then $(x)_K$ is subsequence of $x = (x_{jk})_{j,k\in\mathbb{N}}$ and $l = (1,1)$ is a *limit point* of a double sequence $(x_{jk})_{j,k\in\mathbb{N}}$.

Definition 3.13. Let $x = (x_{jk})_{j,k\in\mathbb{N}}$ be adouble sequence in 2-normed space $(X, \|.,.\|)$. An element $l \in X$ is said to be an I_2*-limit point* of $(x_{jk})_{j,k\in\mathbb{N}}$ if there exists a set $M = \{(m_j, m_k) : j,k \in \mathbb{N}\} \subseteq \mathbb{N}\times\mathbb{N}$ such that $M \notin I$ and $\lim_{m_j,m_k} x_{m_j m_k} = l$

We now introduce the notations L_x^2 and $I(\Lambda_x^2)$ to denote the set of all *limit points* and I*-limit points* of $(x_{jk})_{j,k\in\mathbb{N}}$ respectively.

Example 3.14. If we consider double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ introduced in Example(3.12)and $I = \{A \subset \mathbb{N}\times\mathbb{N} : d_2(A) = 0\}$ then $I(\Lambda_x^2) = \emptyset$.
Otherwise,there exists $M = \{(m_j, m_k) \in \mathbb{N}\times\mathbb{N} : j,k \in \mathbb{N}\}$ such that $d_2(M) > 0$
By definition $x = (x_{jk})_{j,k\in\mathbb{N}}$ we have $m_j \neq m_k \Rightarrow x_{m_j m_k} = (1, m_k)$
and

$$\|x_{m_j,m_k} - \beta\|_\infty = \max\{|1-b_1|, |m_k - b_2|\} \text{ where } \beta = (b_1, b_2).$$

If $m_j = m_k \Rightarrow x_{m_j m_k} = (1,1)$
and

$$\|x_{m_j,m_k} - \beta\|_\infty = \max\{|1-b_1|, |1-b_2|\}$$

On other word $d_2(M) = \lim_{m,n} \frac{M(m,n)}{mn} > 0$
Hence $\lim \|x_{m_j m_k} - \beta\|_\infty \neq 0$.

We show in Example 3.12 and 3.14 that in general, L_x^2 and $I(\Lambda_x^2)$ may be quit deferent.

Definition 3.15. An element $\alpha \in X$ is said to be an I_2-*cluster point* of a double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ if for each $\varepsilon > 0$ and nonzero $z \in X$ the set $\{(j,k) : \|x_{jk} - \alpha, z\| < \varepsilon\} \notin I$.
We denote the set of all I_2-*cluster points* of x by $I(\Gamma_x^2)$.

Theorem 3.16: Let I be a strongly admissible ideal.Then for any double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ in $(X, \|.,.\|)$ we have $I(\Lambda_x^2) \subseteq I(\Gamma_x^2)$.

proof: Let $\alpha \in I(\Lambda_x^2)$. Then there exists a set

$$M = \{(m_i, m_j) \in \mathbb{N} \times \mathbb{N} : j, k \in \mathbb{N}\}$$

such that

$$\lim_{m_j, m_k} \|x_{m_j m_k} - \alpha, z\| = 0 \text{ for each } z \in X. \ (1)$$

Let $\varepsilon > 0$.By(1)there exists $n_0 \in N$ such that:
for all $m_j, m_k \geq n_0$ we have $\|x_{m_j m_k} - \alpha, z\| < \varepsilon$ for each nonzero $z \in X$.
So,for each nonzero $z \in X$ we have
$\{(j,k) : \|x_{jk} - \alpha, z\| < \varepsilon\} \supseteq M \backslash \{(m_j, m_k) : \textbf{\textit{either}}\ m_j \leq n_0 - 1 \ \textbf{\textit{or}}\ m_k \leq n_0 - 1\}$.
Since I is strongly admissible,so

$$\{(j,k) : \|x_{jk} - \alpha, z\| < \varepsilon\} \notin I \text{ for each } z \in X$$

This implies $\alpha \in I(\Gamma_x^2)$,which completes the proof.

Corollary 3.17: Let $(X, \|.,.\|)$ be finite dimensional 2-normed space and $I \subset \mathbb{N} \times \mathbb{N}$ be a admissible ideal.Then for each double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ in $(X, \|.,.\|)$ the set $I(\Gamma_x^2)$is closed in X.
proof: Let $y \in \overline{I(\Gamma_x^2)}$. Put $\varepsilon > 0$ then there exists $l \in I(\Gamma_x^2) \bigcap B_v(y, \varepsilon)$. Choose $\delta > 0$ such that $B_v(l, \delta) \subseteq B_v(y, \varepsilon)$. Hence we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|y - x_{mn}, z\| < \varepsilon\} \supseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|l - x_{mn}, z\| < \delta\} \notin I$$

Therefor $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|y - x_{mn}, z\| < \varepsilon\} \notin I$ and $y \in I(\Gamma_x^2)$.

Corollary 3.18: Let $(X, \|.,.\|)$ be 2-normed space and $\mathcal{M}_2^2$ be the set of all bounded double sequence of X with norm

$$\|x\| = sup_{m,n} \|x_{mn}, z\| \text{ for each } z \in X, \text{ where } x = (x_{mn})_{m,n\in\mathbb{N}}$$

Then $\mathcal{M}_2^2$ is a norm linear space.

Theorem 3.19: Let $(X, \|.,.\|)$ be a 2-Banach space.If I be a nontrivial admissible ideal of $\mathbb{N}\times\mathbb{N}$ and $\mathcal{M}_I^2$ denoted the set all *bounded I_2-convergent double sequences* of X then the set $\mathcal{M}_I^2$ is a closed linear subspace of the norm linear space $\mathcal{M}_2^2$.

proof: From Corollary(3.9) we see that $\mathcal{M}_I^2$ is a linear subspace of $\mathcal{M}_2^2$. Therefor we only show that $\mathcal{M}_I^2$ is closed in $\mathcal{M}_2^2$.
Let $x^p \in \mathcal{M}_I^2$ $(p = 1.2, ...)$ and $\lim_p \|x^p - x, z\| = 0$ for each $z \in X$ and $x \in \mathcal{M}_2^2$.
We clime that $x \in \mathcal{M}_I^2$.
Since $x^p \in \mathcal{M}_I^2$,for each p there exists an element $a_p \in X$ such that

$$I_2\text{-}\lim_{m,n} x_{mn}^p = a_p \ (p = 1, 2, ...) \text{ , where } x^p = (x_{mn})_{m,n\in\mathbb{N}}.$$

We now prove the following statements:
(i)There exists $\mathbf{a} \in X$ such that $a_p \xrightarrow{\|.,.\|_X} \mathbf{a}$.
(ii)I_2-$\lim_{m,n} x_{mn} = \mathbf{a}$, where $x = (x_{mn})_{m,n\in\mathbb{N}}$.
The result will then follows from (i) and (ii).

Proof of (i):
We have $x^p \xrightarrow{\|.,.\|_X} x \in m_2$. Hence, for each $\varepsilon > 0$ and $z \in X$,there exists $n_0 \in \mathbb{N}$ such that, for each $q \geq r \geq N_0$, we have,

$$\|x^q - x^r, z\| < \tfrac{\varepsilon}{3}$$

Now since $x^q, x^r \in m_I^2$, so I_2-$\lim_{m,n} x_{mn}^q = a_q$ and I_2-$\lim_{m,n} x_{mn}^r = a_r$.
Therefore

$$A_q = \{(m, n) \in \mathbb{N}\times\mathbb{N} : \|x^q - a_q, z\| < \tfrac{\varepsilon}{3}\} \in F(I) \text{ for each } z \in X$$
$$A_r = \{(m, n) \in \mathbb{N}\times\mathbb{N} : \|x^r - a_r, z\| < \tfrac{\varepsilon}{3}\} \in F(I) \text{ for each } z \in X$$

Then $A_r \cap A_q \in F(I)$. Since I is nontrivial and admissible so $A_r \cap A_q$ must be infinite set.Choose $(m_0, n_0) \in A_r \cap A_q$ and therefore,for each $z \in X$

$$\|x_{m_0n_0}^q - a_q, z\| < \tfrac{\varepsilon}{3} \text{ and } \|x_{m_0n_0}^r - a_r, z\| < \tfrac{\varepsilon}{3}$$

Hence
for each $z \in X$ and $q \geq r \geq N_0$ we have,

$$\|a_q - a_r, z\| \leq \|a_q - x_{m_0n_0}^q, z\| + \|x_{m_0n_0}^q - x_{m_0n_0}^r, z\| + \|x_{m_0n_0}^r - a_r, z\| < \varepsilon$$

So $(a_p)_{p\in\mathbb{N}}$ is a Cauchy sequence in 2-Banach space X and so it must converge to an element $\mathbf{a} \in X$. Hence $a_p \xrightarrow{\|.,.\|_X} \mathbf{a}$.

Proof of (ii):
Let $\delta > 0$.Since $x^p \xrightarrow{\|.,.\|_X} x$ there exists $q \in \mathbb{N}$ such that

$$\|x^q - x, z\| < \frac{\delta}{3} \qquad \text{for each } z \in X \tag{3.1}$$

The number q can be chosen in such a way that together with (3.1) the inequality $\|a_q - a, z\| < \frac{\delta}{3}$ for each $z \in X$ also holds.

Because I_2-$\lim_{m,n} x_{mn}^{(q)} = a_q$,hence

$$A_q = \{(m,n) \in \mathbb{N} : \|x_{mn}^q - a_q, z\| < \tfrac{\delta}{3}\} \in F(I) \text{ for each } z \in X.$$

Now for each $(m,n) \in A_q$ we have:

$$\|x_{mn} - \mathbf{a}, z\| \leq \|x_{mn} - x_{mn}^{(q)}, z\| + \|x_{mn}^{(q)} - a_q, z\| + \|a_q - \mathbf{a}, z\| < \tfrac{\delta}{3} + \tfrac{\delta}{3} + \tfrac{\delta}{3} < \delta$$

Hence

$$A_q = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - \mathbf{a}, z\| < \delta\} \in F(I) \text{ for each } z \in X$$

This implies that:

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - \mathbf{a}, z\| \geq \delta\} \in I \text{ for each } z \in X$$

Therefore I_2-$\lim_{m,n} x_{mn} = \mathbf{a}$. This completes the proof of the theorem.

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References

1. V.Baláž, J.Cervenansky,P.Kostyrko,T.Salat , *I-convergence and I-continuity of real functions* ,In: Acta Mathematica 5, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, (2004) **43-50**.
2. R.C.Buck, *Generalized asymptotic density*, Amer.J.Math, (1953) **335-336**.
3. Y.J.Cho,P.C.S.Lin,S.S.Kim, A.Misiak, *Theory of 2-inner product spaces*, Nova Science,Huntington,NY,USA,(2001) .
4. J.S. Connor, *The statistical and strong p-Cesaro convergence of sequences*, Analysis , **8** (1988) **74-63**.
5. P. Das,P.Kostyrko,W.Wilczyncki, P. Malik , *I and I^*-convergence of double sequence* ,Math.Slovaca, **58** (2008),No.5, **605-620**.
6. P. Das , P. Malik , *On extermal I-limit points of double sequences* ,Tatra Mt.Math, **40** (2008), **91-102**.
7. H. Fast, *Sur la convergence statistique*, Colloq. Math, **2** (1951) **241-244**.
8. R. Freese and S. Gähler, *Remarks on semi 2-normed spaces*, Math. Nachr, **105** (1982) **151-161**.
9. J.A.Fridy, *On statistical convergence*, Analysis, **105** (1985) **301-313**.
10. J.A.Fridy, *Statistical limit points*,Proc. Amer. Math. Soc, **118** (1993) **1187-1192**.
11. S.Gähler, *2-normed spaces*, Math.Nachr, **28** (1964) **1-43**.
12. S.Gähler, *2-metrische Räumm und ihre topologische struktur*, Math. Nachr, **26** (1963) **115-148**.
13. M. Gürdal , *On ideal convergent sequences in 2-normed Spaces*, Thai. J. Math, **4(1)** (2006) **85-91**.
14. M. Gürdal and S. Pehlivan, *The statistical convergence in 2-Banach Spaces*, Thai. J. Math, **2(1)** (2004) **107-113**.

15. H. Gunawan and Mashadi, *On finite dimensional 2-normed spaces*, Soochow J. Math, **27(3)** (2001) **321-329**.
16. E.Kolk, *The statistical convergence in Banack spaces*, Tartu ÜI.Toimetsed, **928** (1991) **41-52**.
17. P.Kostyrko,T.Salat,W.Wilczyncki , *I-convergence* ,Real Anal. Exchange, **26** (2000/2001) **669-666**.
18. P.Kostyrko, M.Macaj, T.Salat, M.Sleziak , *I-convergence and extremal I-limit points* ,Math. Slovaca, **55** (2005) **443-464**.
19. B.Lahiri,P.Das, *I and I^*-convergence in topological spaces* ,Math. Bohem, **130** (2005) **153-160**.
20. F. Moricz, *Tauberian theorems for Cesaro summable double sequences*, Studia Math, **110** (1994) **83-96**.
21. Mursaleen and Osama H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl,**288** (2003) **223-231**.
22. A. Pringsheim, *Zur Ttheorie der zweifach unendlichen Zahlenfolgen*, Math. Ann, **53** (1900) **289-321**.
23. W. Raymond, Y. Freese , J. Cho, *Geometry of linear 2-normed spaces*, N.Y. Nova Science Publishers, Huntington, 2001.
24. S. Sarabadan , S. Talebi, *Statistical convergence of double sequences in 2- normed spaces* , Int. J. Contemp. Math. Sciences., **6** (2011) **373 - 380**.
25. S. Sarabadan , S. Talebi, *Statistical convergence and ideal convergence of sequences of functions in 2- normed spaces* ,International Journal of Mathematics and Mathematical Sciences, hindawi Publishing Corporation.(**vol(2011)**)
26. **S. Sarabadan , S. Talebi,** *A condition for the equivalence of I- and I^*-convergence in 2- normed spaces* **, Int. J. Contemp. Math. Sciences., 6(no.43) (2011) 2147 - 2159.**
27. **H. Steinhaus,** *Sur la convergence ordinarie et la convergence asymptotique* **, Colloq. Math , 2 (1953) 335-346.**
28. **C.Tripathy,B.C.Tripathy,** *On I-convergent double series* **,Soochow J. Math, 31 (2005) 549-560.**

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