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Iterative Methods for Solving Trifunction Variational Inequalities

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Abstract

In this paper, we introduce and consider a class of variational inequalities, which is called the trifunction variational inequality. We use the auxiliary principle technique to suggest and analyze an implicit iterative methods for solving the trifunction variational inequalities. We study the convergence criteria of this new method under pseudomomotonicity condition, which is weaker condition than monotonocity. Several special cases are also considered.

Mathematics Subject Classification: 47H05; 47H10

Keywords: Variational inequalities, auxiliary principle, convergence iterative methods

1 Introduction

Variational inequalities are being used a mathematical tool to study a wide class of problems which arise in various fields of mathematical and engineering sciences. It is well known that the optimality condition for the differentiable convex function on the convex set can be characterized by the variational inequality. However, it has been shown that the optimality condition of a directionally differentiable convex function can be characterized by a class of variational inequality, which is called the bifunction variational inequality, see [1-6, 12,14,15,17-22]. Motivated and inspired by the on going research in this area, we introduce and consider a new class of variational inequalities which is called the trifunction variational inequalities. As special cases, we obtain the bifunction variational inequalities and other related optimization problems. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving trifunction variational inequalities due to nature of the problem. This fact has motivated to use the auxiliary principle technique, which is mainly due to Glowinski, Lions and Tremolieres [7]. This technique has been used extensively by Noor [9-11] and Noor et al. [23] to suggest and analyze an implicit iterative method for solving variational inequalities and related problems. We use this technique couplid with Bregman function to suggest and an analyze an implicit iterative method for solving the trifunction variational inequalities. We also study the convergence of this new method under the pseudomonotonicity. It is well-known that the pseudomonotonicity implies monotonicity, but the converse is not true. This shows that the pseudomonotonicity is a weaker condition than monotonicity. In this sense, our result represents a refinement of the previous known results. Our results can be considered as a novel and important application of the auxiliary principle technique. We expect that the idea and techniques of this paper may lead to open further research opportunities and applications of the trifunction variatinal inequalities. For the recent developments in this direction, see [12-16, 17-22] and the references therein.

2 Preliminaries and Formulation of Problem

Let H be a real Hilbert space, where inner product and norm are denotee by $\langle ., . \rangle$ and $||.||$ respectively. Let K be a nonempty closed set in H and T be a nonlinear operator.
For a given continuous trifunction function $F : K \times K \times K \rightarrow H$ and the operator T, we consider the problem of finding $u \in K$ such that

$$F(u, Tu, v - u) \geq 0, \quad \forall \quad v \in K, \tag{2.1}$$

which is called an trifunction variational inequality. A number of problems arising in various branches of pure and applied sciences can be studied via the trifunction variational inequalities.
Some special cases:
(i) If $F(u, Tu, v-u) = B(u, v-u)$, where $B(., .) : K \times K \rightarrow H$ is a bifunction, then problem 2.1 is equivalent to finding $u \in K$ such that

$$B(u, v - u) \geq 0, \quad \forall \quad v \in K, \tag{2.2}$$

which is called the bifunction variational inequality. A number of problems arising in various branches of pure and applied sciences can be studied via the bifunction variational inequalities. It has been shown [12] that the minimum of a directionally differetiable convex function on a convex set can be characterized by the bifunction variational inequality (2.2). In a similar way one

can show that the minimum of a Lipschitz continuous nonconvex satisfies is a solution of the bifunctional variational inequality 2.2. For the formulation, well posedness, existence results for bifunction variational inequalities , see [1-6, 12, 17-22] and the references therein.
(ii) If $F(u, Tu, v-u) =< Tu, v-u >$, then the trifunction variational inequality 2.1 is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall \;\; v \in K, \tag{2.3}$$

which is known as the classical variational inequality introduced by Stampacchia [24]. For the recent applications, numerical methods and formulations of variational inequalities, and equilibrium problems, see [1-25] and the references therein.

Definition 2.1 *The trifunction $F(.,.,.)$ and the operator T is said to be jointly pseudomonotone, if and only if,*

$$F(u, Tu, v-u) \geq 0$$
$$\Rightarrow \qquad -F(v, Tv, u-v) \geq 0, \quad \forall \;\; u, v \in K.$$

3 Main Results

In this section, we consider an iterative method for solving the trifunction variational inequality 2.1 by using the technique of the auxiliary technique, which is due to Glowinski et al. [7] as developed by Noor [9-11] and Noor et al. [23].
For a given $u \in K$ satisfying 2.1, we consider the problem of finding $w \in K$ such that

$$\rho F(w, Tw, u-w) + \langle E'(w) - E'(u), v-w\rangle \geq 0, \quad \forall \;\; v \in K, \tag{3.1}$$

which is called the auxiliary trifunction variatinal inequality. Here $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex functin E at $u \in K$. From the strongly convexity of the differentiable function E(u), it follows that problem 3.1 has a unique solution. It is clear that if $w = u$, then w is a solution of problem 2.1. This observation enables to suggest and analyze the following iterative method for solving the problem 2.1
Algorithm 3.1. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall \; v \in K, \tag{3.2}$$

where $\rho > 0$ is a constant.
Algorithm 3.1 is known as the implicit method for solving the trifunction

variational inequality 2.1. If $f(u,Tu,v-u) = B(u,v-u)$, then Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho B(u_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall \ v \in K,$$

for solving the bifunction variational inequalities 2.2, see [14,15,17-22] and the references therein.

Note that, if $F(u,Tu,v-u) = \langle Tu, v-u\rangle$, then Algorithm 3.1 reduces to the following iterative scheme for variational inequalities 2.3 and appears to be a new one.

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall \ v \in K,$$

where $\rho > 0$ is a constant. For appropriate and suitable choice of the bifunction and the spaces, one can obtain a number of iterative methods for solving the bifunction variational inequalities and related optimization problems. We now study the convergence criteria of algorithm 3.1 and this is the main motivation of next result.

Theorem 3.1 *Let the function $F(.,.,.)$ be jointly pseudomonotone with respect to the operator T and let $E(u)$ be strongly convex function with modulus $\beta > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in K$ of the trifunction variational ineuality 2.1.*

Proof *Let $u \in K$ be a solution 2.1. Then, using the jointly pseudomonotonicity of $F(.,.,.)$, we have*

$$-F(u, Tv, u-v) \geq 0, \quad \forall \ v \in K. \tag{3.3}$$

Taking $v = u_{n+1}$ in 3.3 and $v = u$ in 3.2, we have

$$-F(u_{n+1}, Tu_{n+1}, v-u_{n+1}) \geq 0 \tag{3.4}$$

$$\rho F(u_{n+1}, Tu_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \tag{3.5}$$

Now we consider the generalized Bregman function as

$$B(u,z) = E(u) - E(z) - \langle E'(z), u-z\rangle \geq \beta ||u-z||^2, \tag{3.6}$$

where we have used the fact that the function E(u) is strongly convex.
Combining (3.4)- (3.6), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u-u_n\rangle + \langle E'(u_{n+1}), u-u_{n+1}\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u-u_{n+1}\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad - \langle E'(u_n), u_{n+1} - u_n \rangle \\
\geq &\quad \beta ||u_{n+1} - u_n||^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\
\geq &\quad \beta ||u_{n+1} - u_n||^2 - \rho F(u_{n+1}, Tu_{n+1}, u - u_{n+1}) \\
\geq &\quad \beta ||u_{n+1} - u_n||^2.
\end{aligned}
$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of 2.1. Otherwise, for $\beta > 0$, the sequences $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} (||u_{n+1} - u_n||) = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [25], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the trifunction variatinal inequality 2.1.

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