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On Generalized Anti Fuzzy Bi-Ideals in Ordered Γ-Semigroups

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Abstract

In this paper, we introduce the concept of an anti fuzzy bi-ideal of ordered Γ-Semigroups by using the notion of anti fuzzy points and besideness to and non-quasi-coincidence with a fuzzy set, and investigate their inter-relations and related properties.

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1 Introduction

The concept of a fuzzy set was first initiated by Zadeh [5]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. The concept of Γ-semigroups was introduced by Sen [3]. Many fundamental results in semigroup theory have been extended to Γ-semigroups. In this paper, we introduce the concept of an anti fuzzy bi-ideal of ordered Γ-Semigroups by using the notion of anti fuzzy points and besideness to and non-quasi-coincidence with a fuzzy set, and investigate their inter-relations and related properties.

2 Preliminary Notes

Let S and Γ be nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$, written (a, γ, b) by $a\gamma b$, S is called a Γ-semigroup if S satisfies the identities $(a\gamma b)\alpha c = a\gamma(b\alpha c)$ for all $a, b, c \in S$ and $\gamma, \alpha \in \Gamma$.

An ordered Γ-semigroup is a structure $(S; \cdot, \leq)$ satisfying the following condition:
(1) $(S; \cdot)$ is a Γ-semigroup,
(2) $(S; \leq)$ is a poset,
(3) $(\forall a, b, x \in S$ and $\gamma \in \Gamma)(a \leq b \Longrightarrow x\gamma a \leq x\gamma b$ and $a\gamma x \leq b\gamma x)$.

Let $(S; \cdot, \leq)$ be an ordered Γ-Semigroup. For $A \subseteq S$, we denote

$$(A] := \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

For $A, B \subseteq S$, we denote, $A\Gamma B := \{a\gamma b \mid a \in A, \gamma \in \Gamma, b \in B\}$. Let $A, B \subseteq S$. Then $A \subseteq (A]$, $(A]\Gamma(B] \subseteq (A\Gamma B]$ and $((A]] = (A]$.

Let S be an ordered Γ-Semigroup and $\emptyset \neq G \subseteq S$. Then G is called a Γ-*subsemigroup* of S if $G\Gamma G \subseteq G$. A Γ-subsemigroup G of an ordered Γ-Semigroup S is called a *bi-ideal* of S if
(1) $G\Gamma S\Gamma G \subseteq G$,
(2) $(\forall x, y \in S)(\forall y \in G)(x \leq y \Rightarrow x \in G)$.

A fuzzy subset μ of S means that a mapping $\mu : S \to [0, 1]$.

Definition 2.1 *A fuzzy subset μ of S is called an anti fuzzy bi-ideal of S if it satisfies:*
(1) $(\forall x, y \in S)(x \leq y \Rightarrow \mu(x) \leq \mu(y))$,
(2) $(\forall x, y \in T \text{ and } \gamma \in \Gamma)(\mu(x\gamma y) \leq \max\{\mu(x), \mu(y)\})$.

A fuzzy subset μ of S of the form

$$\mu(y) = \begin{cases} t \in [0, 1) & \text{if } y = x, \\ 1 & \text{otherwise,} \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by $\frac{t}{x}$. A fuzzy subset μ of S is said to be non-unit if there exists $x \in S$ such that $\mu(x) < 1$.

For an anti fuzzy point $\frac{t}{x}$ and a fuzzy subset μ in S, Jun and Song [4] gave meaning to the symbol $\frac{t}{x}\alpha\mu$, where $\alpha \in \{[\in], [q], [\in] \vee [q], [\in] \wedge [q]\}$.

To say that $\frac{t}{x}[\in]\mu$(resp. $\frac{t}{x}[q]\mu$) means that $\mu(x) \leq t$ (resp. $\mu(x) + t < 1$), and in this case, $\frac{t}{x}$ is said to be *beside to*(resp. *be non-quasi-coincident with*) a fuzzy subset μ. To say that $\frac{t}{x}[\in] \vee [q]\mu$ (resp. $\frac{t}{x}[\in] \wedge [q]\mu$) means that $\frac{t}{x}[\in]\mu$ or $\frac{t}{x}[q]\mu$ (resp. $\frac{t}{x}[\in]\mu$ and $\frac{t}{x}[q]\mu$). To say that $\frac{t}{x}\overline{\alpha}\mu$ means that $\frac{t}{x}\alpha\mu$ does not hold.

3 Main Results

In what follows let S denote an ordered Γ-Semigroup unless otherwise specified.

Definition 3.1 *A fuzzy subset μ of S is called an $([\in],[\in])$-fuzzy bi-ideal of S if it satisfies:*
(1) $(\forall x, y \in S)(\forall t \in [0,1))\left(x \leq y, \frac{t}{y}[\in]\mu \Rightarrow \frac{t}{x}[\in]\mu\right)$,
(2) $(\forall x, y \in S$ *and* $\gamma \in \Gamma)(\forall t_1, t_2 \in [0,1))\left(\frac{t_1}{x}[\in]\mu, \frac{t_2}{y}[\in]\mu \Rightarrow \frac{\max\{t_1,t_2\}}{x\gamma y}[\in]\mu\right)$,
(3) $(\forall x, y, a \in S$ *and* $\gamma_1, \gamma_2 \in \Gamma)(\forall t_1, t_2 \in [0,1))\left(\frac{t_1}{x}[\in]\mu, \frac{t_2}{y}[\in]\mu \Rightarrow \frac{\max\{t_1,t_2\}}{x\gamma_1 a\gamma_2 y}[\in]\mu\right)$.

Theorem 3.2 *A fuzzy subset μ of S is an $([\in],[\in])$-fuzzy bi-ideal of S if and only if it satisfies:*
(i) $(\forall x, y \in S)(x \leq y \Rightarrow \mu(x) \leq \mu(y))$,
(ii) $(\forall x, y \in S$ *and* $\gamma \in \Gamma)(\mu(x\gamma y) \leq \max\{\mu(x), \mu(y)\})$,
(iii) $(\forall x, y, a \in S$ *and* $\gamma_1, \gamma_2 \in \Gamma)(\mu(x\gamma_1 a\gamma_2 y) \leq \max\{\mu(x), \mu(y)\})$.

Proof. Assume that μ satisfies the conditions (i), (ii) and (iii). Let $x, y, a \in S$, $\gamma_1, \gamma_2 \in \Gamma$ and $t \in [0,1)$ be such that $x \leq y$ and $\frac{t}{y}[\in]\mu$. Using (i), we have $\mu(x) \leq \mu(y) \leq t$, and so $\frac{t}{x}[\in]\mu$. Let $a, x, y \in S$ and $t_1, t_2 \in [0,1)$ be such that $\frac{t_1}{x}[\in]\mu$ and $\frac{t_2}{y}[\in]\mu$. Then $\mu(x) \leq t_1$ and $\mu(y) \leq t_2$, which implies from (ii) and (iii) that

$$\mu(x\gamma_1 y) \leq \max\{\mu(x), \mu(y)\} \leq \max\{t_1, t_2\},$$

$$\mu(x\gamma_1 a\gamma_2 y) \leq \max\{\mu(x), \mu(y)\} \leq \max\{t_1, t_2\}.$$

Hence $\frac{\max\{t_1,t_2\}}{x\gamma_1 y}[\in]\mu$ and $\frac{\max\{t_1,t_2\}}{x\gamma_1 a\gamma_2 y}[\in]\mu$.

Conversely, assume that a fuzzy subset μ of S is an $([\in],[\in])$-fuzzy bi-ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $\mu(y) \leq \mu(x)$, then there exists $t \in (0,1)$ such that $\mu(y) \leq t < \mu(x)$. Thus $\frac{t}{y}[\in]\mu$, but $\frac{t}{x}\overline{[\in]}\mu$. This is impossible, and therefore $\mu(x) \leq \mu(y)$ for all $x, y \in S$ with $x \leq y$. Suppose that (ii) is not valid. Then

$$\max\{\mu(x), \mu(y)\} \leq t < \mu(x\gamma_1 y)$$

for some $x, y \in S$, $\gamma \in \Gamma$ and $t \in (0,1)$. It follows that $\frac{t}{x}[\in]\mu$, and $\frac{t}{y}[\in]\mu$, but $\frac{t}{x\gamma y}\overline{[\in]}\mu$. This is a contradiction. Hence $\mu(x\gamma y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$. Finally assume that (iii) is false. Then

$$\max\{\mu(x), \mu(y)\} \leq t < \mu(x\gamma a\gamma_1 y),$$

for every $a, x, y \in S$, $\gamma, \gamma_1 \in \Gamma$ and $t \in (0,1)$. Hence $\frac{t}{x}[\in]\mu$, and $\frac{t}{y}[\in]\mu$, but $\frac{t}{x\gamma a\gamma_1 y}\overline{[\in]}\mu$. This is a contradiction. Therefore $\mu(x\gamma a\gamma_1 y) \leq \max\{\mu(x), \mu(y)\}$ for all $a, x, y \in S$ and $\gamma, \gamma_1 \in \Gamma$. ∎

Remark 3.3 From Theorem 3.2, it follows that every anti fuzzy bi-ideal of S is an $([\in],[\in])$-fuzzy bi-ideal of S.

Definition 3.4 *A fuzzy subset μ of S is an $([\in],[\in]\vee[q])$-fuzzy bi-ideal of S if and only if it satisfies:*
(i) $(\forall x,y\in S)(\forall t\in[0,1))\left(x\leq y, \frac{t}{y}[\in]\mu\Rightarrow\frac{t}{x}[\in]\vee[q]\mu\right)$,
(ii) $(\forall x,y\in S$ *and* $\gamma\in\Gamma)(\forall t_1,t_2\in[0,1))\left(\frac{t_1}{x}[\in]\mu, \frac{t_2}{y}[\in]\mu\Rightarrow\frac{\max\{t_1,t_2\}}{x\gamma y}[\in]\vee[q]\mu\right)$,
(iii) $(\forall x,y,a\in S$ *and* $\gamma_1,\gamma_2\in\Gamma)(\forall t_1,t_2\in[0,1))\left(\frac{t_1}{x}[\in]\mu, \frac{t_2}{y}[\in]\mu\Rightarrow\frac{\max\{t_1,t_2\}}{x\gamma_1 a\gamma_2 y}[\in]\vee[q]\mu\right)$.

Theorem 3.5 *A fuzzy subset μ of S is an $([\in],[\in]\vee[q])$-fuzzy bi-ideal of S if and only if it satisfies:*
(i) $(\forall x,y\in S)(x\leq y\Rightarrow\mu(x)\leq\max\{\mu(y),0.5\})$,
(ii) $(\forall x,y\in S$ *and* $\gamma\in\Gamma)(\mu(x\gamma y)\leq\max\{\mu(x),\mu(y),0.5\})$,
(iii) $(\forall x,y,a\in S$ *and* $\gamma_1,\gamma_2\in\Gamma)(\mu(x\gamma_1 a\gamma_2 y)\leq\max\{\mu(x),\mu(y),0.5\})$.

Proof. Suppose that μ satisfies the conditions (i), (ii) and (iii). Let $x,y\in S, x\leq y$ and $t\in[0,1)$ be such that $\frac{t}{x}[\in]\mu$. Then $\mu\leq t$, by using (i), we have

$$\mu(x)\leq\max\{\mu(y),0.5\}\leq\max\{t,0.5\}.$$

Thus $\mu(x)\leq t$ or $\mu(x)\leq 0.5$, according to $t>0.5$ or $t\leq 0.5$. Thus $\frac{t}{x}[\in]\vee[q]\mu$. Let $x,y\in S$, $\gamma\in\Gamma$ and $t,r\in[0,1)$ be such that $\frac{t}{x}[\in]\mu$ and $\frac{r}{y}[\in]\mu$. Then $\mu(x)\leq t$ and $\mu(y)\leq r$ and by using (ii) we have

$$\mu(x\gamma y)\leq\max\{\mu(x),\mu(y),0.5\}\geq\max\{t,r,0.5\},$$

If $\max\{t,r\}<0.5$ then $\mu(x\gamma y)\leq 0.5$ and so $\mu(x\gamma y)+\max\{t,r\}<0.5+0.5=1$, and we have $\frac{\max\{t,r\}}{x\gamma y}[q]\mu$. If $\max\{t,r\}\geq 0.5$ then $\mu(x\gamma y)\leq\max\{\mu(x),\mu(y)\}$ and so $\frac{\max\{t,r\}}{x\gamma y}[\in]\mu$ and hence $\frac{\max\{t,r\}}{x\gamma y}[\in]\vee[q]\mu$. Let $a,x,y\in S$, $\gamma_1,\gamma_2\in\Gamma$ and $t,r\in(0,1]$ be such that $\frac{t}{x}[\in]\mu$ and $\frac{r}{y}[\in]\mu$. Then $\mu(x)\leq t$ and $\mu(y)\leq r$ and bu using (iii) we have

$$\mu(x\gamma_1 a\gamma_2 y)\leq\max\{\mu(x),\mu(y),0.5\}\geq\max\{t,r,0.5\}.$$

If $\max\{t,r\}<0.5$ then $\mu(x\gamma_1 a\gamma_2 y)\leq 0.5$ and so $\mu(x\gamma_1 a\gamma_2 y)+\max\{t,r\}<0.5+0.5=1$, and we have $\frac{\max\{t,r\}}{x\gamma_1 a\gamma_2 y}[q]\mu$. If $\max\{t,r\}\geq 0.5$ then $\mu(x\gamma_1 a\gamma_2 y)\leq\max\{\mu(x),\mu(y)\}$ and so $\frac{\max\{t,r\}}{x\gamma_1 a\gamma_2 y}[\in]\vee[q]\mu$.

Conversely, let $x,y\in S$ and $x\leq y$. We consider the following cases:
a) $\mu(y)>0.5$,
b) $\mu(y)\leq 0.5$.

Case a): Let $x,y\in S$ and $x\leq y$. Assume that $\max\{\mu(y),0.5\}<\mu(x)$, which implies that $\mu(y)<\mu(x)$. Choose t such that $\mu(y)\leq t<\mu(x)$. Then $\frac{t}{y}[\in]\mu$ but $\frac{t}{x}\overline{[\in]}\mu$ and so $\frac{t}{x}[\in]\overline{\vee}[q]\mu$. This is a contradiction.

Case b): Let $x, y \in S$ be such that $\max\{\mu(x), \mu(y), 0.5\} < \mu(x\gamma y)$. Then $\mu(x\gamma y) > 0.5$, $\frac{0.5}{x}[\in]\mu$ and $\frac{0.5}{y}[\in]\mu$ but $\frac{0.5}{x}\overline{[\in]}\mu$ and so $\frac{0.5}{x}\overline{[\in] \vee [q]}\mu$. This is a contradiction. Hence $\mu(x) \leq \max\{\mu(y), 0.5\}$ for all $x, y \in S$ with $x \leq y$.

Let $x, y \in S$ and we consider the following cases:
a) $\max\{\mu(x), \mu(y)\} \leq 0.5$,
b) $\max\{\mu(x), \mu(y)\} > 0.5$.

Case a): Let $x, y \in S$ and $\gamma \in \Gamma$ be such that $\max\{\mu(x), \mu(y), 0.5\} < \mu(x\gamma y)$. Then $\mu(x\gamma y) > 0.5$, $\frac{0.5}{x}[\in]\mu$ and $\frac{0.5}{y}[\in]\mu$ but $\frac{0.5}{x\gamma y}\overline{[\in] \vee [q]}\mu$. This is a contradiction.

Case b): Let $x, y \in S$ and $\gamma \in Gamma$ be such that $\max\{\mu(x), \mu(y), 0.5\} < \mu(x\gamma y)$. Then $\max\{\mu(x), \mu(y)\} < \mu(x\gamma y)$. Choose $r \in (0, 1]$ such that $\max\{\mu(x), \mu(y), 0.5\} \leq r < \mu(x\gamma y)$. Then $\frac{r}{x}[\in]\mu$ and $\frac{r}{y}[\in]\mu$ but $\frac{r}{x\gamma y}\overline{[\in]}\mu$ and so $\frac{r}{x\gamma y}\overline{[\in] \vee [q]}\mu$. This is a contradiction. Hence $\mu(x\gamma y) \leq \max\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Let $a, x, y \in S$, $\gamma_1, \gamma_2 \in \Gamma$ and we consider the following cases:
a) $\max\{\mu(x), \mu(y)\} \leq 0.5$,
b) $\max\{\mu(x), \mu(y)\} > 0.5$.

Case a): Let $a, x, y \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ be such that $\max\{\mu(x), \mu(y), 0.5\} < \mu(x\gamma_1 a\gamma_2 y)$. Then $\mu(x\gamma_1 a\gamma_2 y) > 0.5$, $\frac{0.5}{x}[\in]\mu$ and $\frac{0.5}{y}[\in]\mu$ but $\frac{0.5}{x\gamma_1 a\gamma_2 y}\overline{[\in]}\mu$ and $\frac{0.5}{x\gamma_1 a\gamma_2 y}\overline{[\in] \vee [q]}\mu$. This is a contradiction.

Case b): Let $a, x, y \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ be such that $\max\{\mu(x), \mu(y), 0.5\} < \mu(x\gamma_1 a\gamma_2 y)$. Then $\max\{\mu(x), \mu(y)\} < \mu(x\gamma_1 a\gamma_2 y)$. Choose $t \in [0, 1)$ such that $\max\{\mu(x), \mu(y), 0.5\} \leq t < \mu(x\gamma_1 a\gamma_2 y)$. Then $\frac{r}{x}[\in]\mu$ and $\frac{r}{y}[\in]\mu$ but $\frac{r}{x\gamma_1 a\gamma_2 y}\overline{[\in]}\mu$ and $\frac{r}{x\gamma_1 a\gamma_2 y}\overline{[\in] \vee [q]}\mu$. This is a contradiction. Hence $\mu(x\gamma_1 a\gamma_2 y) \leq \max\{\mu(x), \mu(y), 0.5\}$ for all $a, x, y \in S$ and $\gamma_1, \gamma_2 \in \Gamma$. ■

Remark 3.6 From Theorem 3.5 and Remark 3.3, it follows that every anti fuzzy bi-ideal of S is an $([\in], [\in] \vee [q])$-fuzzy bi-ideal of S.

Theorem 3.7 *Let μ be an $([\in], [\in]\vee[q])$-fuzzy bi-ideal of S such that $\mu(x) > 0.5$ for all $x \in S$. Then μ is an $([\in], [\in])$-fuzzy bi-ideal of S.*

Proof. Let $x, y \in S, x \leq y$ and $t \in [0, 1)$ be such that $\frac{t}{y}[\in]\mu$. Then $\mu(y) \leq t$ and by (i) of Theorem 3.5, it follows that

$$\mu(x) \leq \max\{\mu(y), 0.5\} \leq \mu(y) \leq t,$$

and so $\frac{t}{x}[\in]\mu$. Let $a, x, y \in S$, $\gamma_1, \gamma_2 \in \Gamma$ and $r, s \in [0, 1)$ be such that $\frac{r}{x}[\in]\mu$, and $\frac{s}{x}[\in]\mu$. Then $\mu(x) \leq r$ and $\mu(y) \leq s$. By (ii) and (iii) of Theorem 3.5, it follows that

$$\mu(x\gamma_1 y) \leq \max\{\mu(x), \mu(y), 0.5\} \leq \max\{r, s\},$$

$$\mu(x\gamma_1 a\gamma_2 y) \leq \max\{\mu(x), \mu(y), 0.5\} \leq \max\{r, s\}.$$

Hence $\frac{\max\{r,s\}}{x\gamma_1 y}[\in]\mu$ and $\frac{\max\{r,s\}}{x\gamma_1 a\gamma_2 y}[\in]\mu$ and therefore μ is an $([\in],[\in])$-fuzzy bi-ideal of S. ■

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