Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 13, 631 - 640

The Properties of $(L, \odot)$-Filters

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Abstract

In this paper, we define the products of $(L, \odot)$-fuzzy topologies and $(L, \odot)$-filters on strictly two-sided, commutative quantale lattices $(L, \odot)$ and $(L, *)$. Furthermore, we study the convergence of $(L, \odot)$-filters.

Mathematics Subject Classification: 03E72, 54A40, 54B10

Keywords: Quantale lattice, $(L, \odot)$-fuzzy topologies, $(L, \odot)$-filters, $(\mathcal{F}^x, *)$-neighborhood filter

1 Introduction

Höhle and Šostak [6] introduced the notion of $(L, \wedge)$-fuzzy topological spaces on a complete quasi-monoidal lattice (or GL-monoid) instead of a completely distributive lattice or an unit interval. Höhle and Sostak [6] introduced the concept of L-filters for a complete quasi-monoidal lattice L.

In this paper, we define the products of $(L, \odot)$-fuzzy topologies and $(L, \odot)$-filters on strictly two-sided, commutative quantale lattices $(L, \odot)$ and $(L, *)$. Furthermore, we study the convergence of $(L, \odot)$-filters. We define an L-fuzzy neighborhood filter, fuzzy cluster points and fuzzy limit points. Also, we study some properties of them and give examples.

2 Preliminaries

Definition 2.1 [8] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \top, \bot)$ is a complete lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;

(L3) $a = a \odot \top$, for each $a \in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i)\odot b = \bigvee_{i\in\Gamma}(a_i\odot b).$$

Example 2.2 [8] (1) Each frame is a stsc-quantale. In particular, the unit interval $([0,1],\leq,\vee,\wedge,0,1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $([0,1],\leq,t)$, is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

(4) Define a binary operation $\odot$ on $[0,1]$ by $x\odot y = \max\{0, x+y-1\}$. Then $([0,1],\leq,\odot)$ is a stsc-quantale.

Definition 2.3 [6,8] A mapping $\tau : L^X \to L$ is called an $(L,\odot)$-*fuzzy topology* on X if it satisfies the following conditions:

(T1) $\tau(1_\emptyset) = \top$ and $\tau(1_X) = \top$ where $1_\emptyset(x) = \bot$ and $1_X(x) = \top$ for all $x \in X$,

(T2) $\tau(f\odot g) \geq \tau(f)\odot\tau(g)$, for each $f, g \in L^X$,

(T3) $\tau(\bigvee_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma}\tau(f_i)$.

An $(L,\odot)$-fuzzy topology is called *enriched* if

(S) $\tau(\alpha\odot f) \geq \tau(f)$ for each $f \in L^X$ and $\alpha \in L$.

The pair (X,τ) is called an (resp. enriched)$(L,\odot)$-*fuzzy topological space*. Let τ_1 and τ_2 be two $(L,\odot)$-fuzzy topologies on X. τ_1 is called *finer* than τ_2 if $\tau_2(f) \leq \tau_1(f)$ for all $f \in L^X$.

Definition 2.4 [6,8] A mapping $\mathcal{F} : L^X \to L$ is called an $(L,\odot)$-*filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(1_\emptyset) = \bot$ and $\mathcal{F}(1_X) = \top$,

(F2) $\mathcal{F}(f\odot g) \geq \mathcal{F}(f)\odot\mathcal{F}(g)$, for each $f, g \in L^X$,

(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.

An $(L,\odot)$-filter is called *stratified* if

(S) $\mathcal{F}(\alpha\odot f) \geq \alpha\odot\mathcal{F}(f)$ for each $f \in L^X$ and $\alpha \in L$.

The pair $(X,\mathcal{F})$ is called an (resp. a stratified)$(L,\odot)$-*filter space*. $\mathcal{F}_1$ is called *finer* than $\mathcal{F}_2$ if $\mathcal{F}_2(f) \leq \mathcal{F}_1(f)$ for all $f \in L^X$.

Example 2.5 (1) Define a map $[x] : L^X \to L$ as $[x](f) = f(x)$. Then $[x]$ is a stratified $(L,\odot)$-filter on X.

(2) Define a map $\inf : L^X \to L$ as $\inf(f) = \bigwedge_{x\in X} f(x)$. Then $\inf$ is a stratified $(L,\odot)$-filter on X.

Definition 2.6 [6] Let $(L,*)$ and $(L,\odot)$ be stsc-quantales. An operation $\odot$ dominates $*$ if it satisfies:

$$(x_1 * y_1)\odot(x_2 * y_2) \geq (x_1\odot x_2) * (y_1\odot y_2).$$

Example 2.7 (1) For any left-continuous t-norm $*$, $\wedge$ dominates $*$ because

$$(x_1 * y_1) \wedge (x_2 * y_2) \geq (x_1 \wedge x_2) * (y_1 \wedge y_2).$$

(2) Define t-norms as $x \odot y = \frac{xy}{x+y-xy}$ and $x * y = xy$. Then $\odot$ dominates $*$.

Lemma 2.8 [9] *Let* $(L, \odot)$ *and* $(L, *)$ *be stsc-quantales which induce two implications* $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ *and* $a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}$*, respectively. Let* $\odot$ *dominates* $*$*. For each* $a, b, c, a_i, b_i \in L$*, we have the following properties.*

(1) *If* $b \leq c$*, then* $a \odot b \leq a \odot c$ *and* $a * b \leq a * c$*.*
(2) $a \odot b \leq c$ *iff* $a \leq b \to c$*. Moreover,* $a * b \leq c$ *iff* $a \leq b \Rightarrow c$*.*
(3) *If* $b \leq c$*, then* $a \to b \leq a \to c$ *and* $c \to a \leq b \to a$ *for* $\to \in \{\to, \Rightarrow\}$*.*
(4) $a * b \leq a \odot b$*,* $a \to b \leq a \Rightarrow b$ *and* $a * (b \odot c) \leq (a * b) \odot c$*.*
(5) $(a \Rightarrow b) \odot (c \Rightarrow d) \leq (a \odot c) \Rightarrow (b \odot d)$*.*
(6) $(b \Rightarrow c) \leq (a \odot b) \Rightarrow (a \odot c)$*.*
(7) $(b \to c) \leq (a \Rightarrow b) \to (a \Rightarrow c)$ *and* $(b \Rightarrow a) \leq (a \to c) \to (b \Rightarrow c)$
(8) $a_i \to b_i \leq (\bigwedge_{i \in \Gamma} a_i) \to (\bigwedge_{i \in \Gamma} b_i)$*.*
(9) $a_i \to b_i \leq (\bigvee_{i \in \Gamma} a_i) \to (\bigvee_{i \in \Gamma} b_i)$*.*
(10) $(c \Rightarrow a) * (b \to d) \leq (a \to b) \to (c \Rightarrow d)$*.*

Theorem 2.9 [9] *Let* $F = \{\mathcal{F}^x \mid x \in X\}$ *be a family of* $(L, \odot)$*-filters for each* $x \in X$*. An operation* $\odot$ *dominates* $*$ *which induces* $a \Rightarrow b = \bigvee\{c \in L \mid a * c \leq b\}$*. For each* $x \in X$*,* $H^x : L^X \to L$ *is a map satisfying the following conditions:*

(H1) $H^x(\top) = \top, H^x(\bot) = \bot$*.*
(H2) $H^x(f \odot g) \leq H^x(f) \odot H^x(g)$*.*
(H3) $H^x(\bigvee f_i) \leq \bigvee H^x(f_i)$*.*

We define a map $\tau_F : L^X \to L$ *as follows:*

$$\tau_F(f) = \bigwedge_{x \in X} (H^x(f) \Rightarrow \mathcal{F}^x(f)).$$

Then (1) τ_F *is an* $(L, \odot)$*-fuzzy topology .*

(2) *If* $\mathcal{F}^x$ *is a stratified* $(L, \odot)$*-filter and* $H^x(\alpha \odot g) \leq \alpha \odot H^x(g)$ *for each* $x \in X$*, then* τ_F *is an enriched* $(L, \odot)$*- topology.*

Theorem 2.10 [10] *Let* (X, τ) *be an* $(L, \odot)$*-fuzzy topology and* $\{\mathcal{F}^x \mid x \in X\}$ *a family of* $(L, \odot)$*-filters. An operation* $*$ *dominates* $\odot$*. We define a map* $\mathcal{N}^x_\tau : L^X \to L$ *as follows:*

$$\mathcal{N}^x_\tau(f) = \bigvee_{g \leq f} (\mathcal{F}^x(g) * \tau(g))$$

Then

(1) $\mathcal{N}_\tau^x$ *is an* $(L,\odot)$*-filter.*

(2) *If* $\mathcal{F}^x$ *is a stratified* $(L,\odot)$*-filter and* τ *is an enriched* $(L,\odot)$*-fuzzy topology , then* $\mathcal{N}_\tau^x$ *is a stratified* $(L,\odot)$*-filter*

(3) *If* $\mathcal{F}^x \geq H^x$*, then* $\tau_{\mathcal{N}_\tau^x} \geq \tau$.

(4) *If* $\mathcal{F}^x \leq H^x$*, then* $\mathcal{N}_{\tau_F}^x \leq \mathcal{F}^x$.

Definition 2.11 [10] In above theorem, a map $\mathcal{N}_\tau^x : L^X \to L$ is called $(\mathcal{F}^x,*)$-neighborhood filter induced by $\mathcal{F}^x,\tau$ and operation $*$. A family $\{\mathcal{N}_\tau^x \mid x \in X\}$ is called $(\mathcal{F}^x,*)$-neighborhood system.

3 The products of $(L,\odot)$-filters

Theorem 3.1 *An operation* $*$ *dominates* $\odot$*. Let* $F = \{\mathcal{F}^x \in L^{L^X} \mid x \in X\}$ *and* $G = \{\mathcal{G}^x \in L^{L^X} \mid x \in X\}$ *be two families of* $(L,\odot)$*-filters satisfying the condition* $\mathcal{F}^x(f) * \mathcal{G}^x(g) = \bot$ *for each* $f \odot g = \bot$*. We define* $\mathcal{F}^x * \mathcal{G}^x : L^X \to L$ *as follows:*

$$\mathcal{F}^x * \mathcal{G}^x(h) = \bigvee\{\mathcal{F}^x(f) * \mathcal{G}^x(g) \mid f \odot g \leq h\}.$$

Let τ_1,τ_2 *be an* $(L,\odot)$*-fuzzy topologies on* X*. We define* $\tau_1 * \tau_2 : L^X \to L$ *as follows:*

$$(\tau_1 * \tau_2)(h) = \bigvee\{\tau_1(f) * \tau_2(g) \mid f \odot g = h\}.$$

(1) $\mathcal{F}^x * \mathcal{G}^x$ *is an* $(L,\odot)$*-filter on* X *which is finer than* $\mathcal{F}^x$ *and* $\mathcal{G}^x$*. If* $* = \odot$*, then* $\mathcal{F}^x \odot \mathcal{G}^x$ *is the coarsest* $(L,\odot)$*-filter on* X *which is finer than* $\mathcal{F}^x$ *and* $\mathcal{G}^x$*. Moreover, if* $* = \odot$ *and* $\mathcal{F}^x = \mathcal{G}^x$*, then* $\mathcal{F}^x \odot \mathcal{F}^x = \mathcal{F}^x$.

(2) *If* $\mathcal{F}^x$ *or* $\mathcal{G}^x$ *is a stratified* $(L,\odot)$*-filter, then* $\mathcal{F}^x * \mathcal{G}^x$ *is a stratified* $(L,\odot)$*-filter on* X.

(3) $\tau_1 * \tau_2$ *is an* $(L,\odot)$*-fuzzy topology on* X *which is finer than* τ_1 *and* τ_2*. If* $* = \odot$*, then* $\tau_1 \odot \tau_2$ *is the coarsest* $(L,\odot)$*-fuzzy topology on* X *which is finer than* τ_1 *and* τ_2.

(4) *If* τ_1 *or* τ_2 *is an enriched* $(L,\odot)$*-fuzzy topology, then* $\tau_1 * \tau_2$ *is an enriched* $(L,\odot)$*-fuzzy topology on* X.

(5) $\tau_{F*G} \geq \tau_F * \tau_G$ *where* $F * G = \{\mathcal{F}^x * \mathcal{G}^x \in L^{L^X} \mid x \in X\}$.

(6) $\mathcal{N}_{\tau_1 \odot \tau_2}^x \geq \mathcal{N}_{\tau_1}^x \odot \mathcal{N}_{\tau_2}^x$.

Proof.(1)

$(\mathcal{F}^x * \mathcal{G}^x)(f) \odot (\mathcal{F}^x * \mathcal{G}^x)(g)$
$= \bigvee\{\mathcal{F}^x(f_1) * \mathcal{G}^x(f_2) \mid f_1 \odot f_2 \leq f\} \odot \bigvee\{\mathcal{F}^x(g_1) * \mathcal{G}^x(g_2) \mid g_1 \odot g_2 \leq g\}$
$= \bigvee\{(\mathcal{F}^x(f_1) * \mathcal{G}^x(f_2)) \odot (\mathcal{F}^x(g_1) * \mathcal{G}^x(g_2)) \mid f_1 \odot f_2 \leq f,\ g_1 \odot g_2 \leq g\}$
(since $*$ dominates $\odot$)
$\leq \bigvee\{(\mathcal{F}^x(f_1) \odot \mathcal{F}^x(g_1)) * (\mathcal{G}^x(f_2) \odot \mathcal{G}^x(g_2)) \mid f_1 \odot f_2 \leq f,\ g_1 \odot g_2 \leq g\}$
$\leq \bigvee\{\mathcal{F}^x(f_1 \odot g_1) * \mathcal{G}^x(f_2 \odot g_2) \mid f_1 \odot f_2 \odot g_1 \odot g_2 \leq f \odot g\}$
$\leq (\mathcal{F}^x * \mathcal{G}^x)(f \odot g)$.

For $f=1_X\odot f$, $\mathcal{F}^x*\mathcal{G}^x\geq \mathcal{F}^x,\mathcal{G}^x$. If $\mathcal{F}^x\leq\mathcal{H}$ and $\mathcal{G}^x\leq\mathcal{H}$, we have $\mathcal{F}^x\odot\mathcal{G}^x\leq\mathcal{H}$. Moreover, $\mathcal{F}^x\odot\mathcal{F}^x=\mathcal{F}^x$ because

$$\mathcal{F}^x\odot\mathcal{F}^x(h)=\bigvee_{f\odot g\leq h}(\mathcal{F}^x(f)\odot\mathcal{F}^x(g))\leq\bigvee_{f\odot g\leq h}\mathcal{F}^x(f\odot g)\leq\mathcal{F}^x(h).$$

Hence the results hold.

(2) Let $\mathcal{F}^x$ be a stratified $(L,\odot)$-filter. Then

$$\begin{aligned}
&\alpha\odot(\mathcal{F}^x*\mathcal{G}^x)(h)\\
&=\alpha\odot\bigvee\{\mathcal{F}^x(f)*\mathcal{G}^x(g)\mid f\odot g\leq h\}\\
&=\bigvee\{\alpha\odot(\mathcal{F}^x(f)*\mathcal{G}^x(g))\mid f\odot g\leq h\}\\
&\leq\bigvee\{(\alpha\odot\mathcal{F}^x(f))*\mathcal{G}^x(g)\mid f\odot g\leq h\}\text{ (by Lemma 2.8(4))}\\
&\leq\bigvee\{\mathcal{F}^x(\alpha\odot f)*\mathcal{G}^x(g)\mid \alpha\odot f\odot g\leq\alpha\odot h\}\\
&\leq(\mathcal{F}^x*\mathcal{G}^x)(\alpha\odot h).
\end{aligned}$$

(3) It is similarly proved as in (1).

(4)

$$\begin{aligned}
&(\tau_1*\tau_2)(\alpha\odot h)\\
&=\bigvee\{\tau_1(h_1)*\tau_2(h_2)\mid h_1\odot h_2=\alpha\odot h\}\\
&\geq\bigvee\{\tau_1(\alpha\odot f)*\tau_2(g)\mid \alpha\odot f\odot g=\alpha\odot h\}\\
&\geq\bigvee\{\tau_1(f)*\tau_2(g)\mid \alpha\odot f\odot g=h\}\\
&=(\tau_1*\tau_2)(h).
\end{aligned}$$

(5) Since $H^x(f\odot g)\leq H^x(f)\odot H^x(g)\leq H^x(f)*H^x(g)$ and $a\to b=\bigvee\{c\mid a\odot c\leq b\}$, by Theorem 2.9, we have

$$\begin{aligned}
\tau_{F*G}(h)&=\bigwedge_{x\in X}(H^x(h)\to\mathcal{F}^x*\mathcal{G}^x(h))\\
&\geq\bigwedge_{x\in X}(\bigvee_{f\odot g\leq h}H^x(f\odot g)\to\bigvee_{f\odot g\leq h}(\mathcal{F}^x(f)*\mathcal{G}^x(g)))\\
&\geq\bigwedge_{x\in X}(\bigvee_{f\odot g\leq h}(H^x(f)\odot H^x(g))\to\bigvee_{f\odot g\leq h}(\mathcal{F}^x(f)*\mathcal{G}^x(g)))\\
&\geq\bigwedge_{x\in X}(\bigvee_{f\odot g\leq h}(H^x(f)*H^x(g))\to\bigvee_{f\odot g\leq h}(\mathcal{F}^x(f)*\mathcal{G}^x(g)))\\
&\quad\text{(by Lemma 2.8(5))}\\
&\geq\bigwedge_{x\in X}\bigvee_{f\odot g\leq h}((H^x(f)\to\mathcal{F}^x(f))*(H^x(g)\to\mathcal{G}^x(g)))\\
&\geq\bigvee_{f\odot g\leq h}\Big(\bigwedge_{x\in X}(H^x(f)\to\mathcal{F}^x(f))*\bigwedge_{x\in X}(H^x(g)\to\mathcal{G}^x(g))\Big)\\
&=(\tau_F*\tau_G)(h).
\end{aligned}$$

(6)

$$\begin{aligned}
&(\mathcal{N}^x_{\tau_1}\odot\mathcal{N}^x_{\tau_2})(h)\\
&=\bigvee\{(\mathcal{N}^x_{\tau_1}(f)\odot\mathcal{N}^x_{\tau_2}(g)\mid f\odot g\leq h\}\\
&=\bigvee\{(\bigvee_{f_1\leq f}\mathcal{F}^x(f_1)*\tau_1(f_1))\odot(\bigvee_{g_1\leq g}\mathcal{F}^x(g_1)*\tau_2(g_1))\mid f\odot g\leq h\}\\
&\leq\bigvee\{(\bigvee_{f_1\odot g_1\leq f\odot g}\mathcal{F}^x(f_1)\odot\mathcal{F}^x(g_1)*(\tau_1(f_1)\odot\tau_2(g_1))\mid f\odot g\leq h\}\\
&\leq\bigvee\{(\bigvee_{f_1\odot g_1\leq f\odot g}\mathcal{F}^x(f_1\odot g_1)*(\tau_1(f_1)\odot\tau_2(g_1))\mid f\odot g\leq h\}\\
&\leq\bigvee\{(\bigvee\mathcal{F}^x(f\odot g)*(\tau_1\odot\tau_2)(f\odot g)\mid f\odot g\leq h\}\\
&\leq\mathcal{N}^x_{\tau_1\odot\tau_2}(h).
\end{aligned}$$

Example 3.2 Let $X = \{x, y\}$ be a set, $(L = [0,1], \odot)$ the stsc-quantale with $x \odot y = 0 \vee (x+y-1)$ and let $f, h \in [0,1]^X$ defined as $f(x) = 0.6, f(y) = 0.5$ and $h(x) = 0.5, h(y) = 0.2$. We define $([0,1], \odot)$-fuzzy topologies $\tau : [0,1]^X \to [0,1]$ as follows:

$$\tau_1(g) = \begin{cases} 1, & \text{if } g \in \{1_X, 1_\emptyset\}, \\ 0.6, & \text{if } g = f, \\ 0.3, & \text{if } g = f \odot f, \\ 0, & \text{otherwise.} \end{cases} \quad \tau_2(g) = \begin{cases} 1, & \text{if } g \in \{1_X, 1_\emptyset\}, \\ 0.5, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases}$$

(1) If $* = \odot$, then we obtain $\tau_1 \odot \tau_2$ as follows:

$$(\tau_1 \odot \tau_2)(g) = \begin{cases} 1, & \text{if } g \in \{1_X, 1_\emptyset\}, \\ 0.6, & \text{if } g = f, \\ 0.5, & \text{if } g = h, \\ 0.3, & \text{if } g = f \odot f, \\ 0.1, & \text{if } g = h \odot f, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}^x(g) = H^x(g) = [x](g) = g(x)$ be given. Then $\mathcal{N}^x_{\tau_i}(f) = \bigvee_{g \le f}(g(x) \odot \tau_i(g))$ for $i = \{1, 2\}$. We obtain $([x], \odot)$ and $([y], \odot)$-neighborhood filters $\mathcal{N}^x_{\tau_i}, \mathcal{N}^y_{\tau_i} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^x_{\tau_1}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.2, & \text{if } f \le g \ne 1_X, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_1}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.1, & \text{if } f \le g \ne 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{N}^x_{\tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{N}^x_{\tau_1 \odot \tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.2, & \text{if } f \le g \ne 1_X, \\ 0.1, & \text{if } h \le g \not\ge f, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_1 \odot \tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.1, & \text{if } f \le g \ne 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{N}^x_{\tau_1 \odot \tau_2} \ge \mathcal{N}^x_{\tau_1} \odot \mathcal{N}^x_{\tau_2}$ and $\mathcal{N}^y_{\tau_1 \odot \tau_2} = \mathcal{N}^y_{\tau_1} \odot \mathcal{N}^y_{\tau_2}$.

(2) If $* = \wedge$, then we obtain $\tau_1 \wedge \tau_2$ as follows:

$$(\tau_1 \wedge \tau_2)(g) = \begin{cases} 1, & \text{if } g \in \{1_X, 1_\emptyset\}, \\ 0.6, & \text{if } g = f, \\ 0.5, & \text{if } g \in \{h, h \odot f\}, \\ 0.4, & \text{if } g = f \odot f, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}^x(g) = H^x(g) = [x](g) = g(x)$ be given. Then $\mathcal{N}^x_{\tau_i}(f) = \bigvee_{g\le f}(g(x)\wedge\tau_i(g))$ for $i = \{1,2\}$. We obtain $([x],\wedge)$ and $([y],\wedge)$-neighborhood filters $\mathcal{N}^x_{\tau_i}, \mathcal{N}^y_{\tau_i} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^x_{\tau_1}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.6, & \text{if } f \le g \ne 1_X, \\ 0.3, & \text{if } f\odot f \le g \not\ge f, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_1}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.5, & \text{if } f \le g \ne 1_X, \\ 0.3, & \text{if } f\odot f \le g \not\ge f, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{N}^x_{\tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.5, & \text{if } h \le g \ne 1_X, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}^y_{\tau_2}(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.2, & \text{if } h \le g \ne 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

For $(f\odot f)\odot h = \overline{0}$, we have $\mathcal{N}^x_{\tau_1}(f\odot f)\wedge\mathcal{N}^x_{\tau_2}(h) = 0.3$ and $\mathcal{N}^y_{\tau_1}(f\odot f)\wedge\mathcal{N}^y_{\tau_2}(h) = 0.2$. Hence $\mathcal{N}^x_{\tau_1}\wedge\mathcal{N}^x_{\tau_2}$ and $\mathcal{N}^y_{\tau_1}\wedge\mathcal{N}^y_{\tau_2}$ do not exist from the condition of Theorem 3.1.

4 $(L,\odot)$-filter convergence

Definition 4.1 Let (X,τ) be an $(L,\odot)$-fuzzy topological space, $\mathcal{N}^x_\tau$ be $(\mathcal{F}^x,*)$-neighborhood filter, $\mathcal{G}$ an $(L,\odot)$-filter , $f,g\in L^X$ and $x\in X$.

(1) x is called $(\mathcal{F}^x,*)$-*cluster point* of $\mathcal{G}$, denoted by $\mathcal{G}\infty x(\mathcal{F}^x,*)$, if for every $\mathcal{N}^x_\tau(f)*\mathcal{G}(g)\ne\bot$, we have $f\odot g\ne 1_\emptyset$.

(2) x is called $(\mathcal{F}^x,*)$-*limit point* of $\mathcal{G}$, denoted by $\mathcal{G}\to x(\mathcal{F}^x,*)$, if $\mathcal{N}^x_\tau\le\mathcal{G}$.

We denote

$$clu_\tau(\mathcal{G})(\mathcal{F}^x,*) = \bigcup\{x\in X \mid x \text{ is } (\mathcal{F}^x,*)\text{-cluster point of } \mathcal{G}\},$$

$$lim_\tau(\mathcal{G})(\mathcal{F}^x,*) = \bigcup\{x\in X \mid x \text{ is } (\mathcal{F}^x,*)\text{-limit point of } \mathcal{G}\}.$$

Theorem 4.2 *Let (X,τ) be an $(L,\odot)$-fuzzy topological space and $\mathcal{N}^x_\tau$ be $(\mathcal{F}^x,*)$-neighborhood filter. Let $\mathcal{F}$ and $\mathcal{G}$ be $(L,\odot)$-filters on X which $\mathcal{F}$ is coarser than $\mathcal{G}$. For each $x\in X$, the following properties hold:*

(1) $\mathcal{N}^x_\tau(f)\to\mathcal{F}(f)\le\mathcal{N}^x_\tau(f)\to\mathcal{G}(f)$, for all $f\in L^X$.

(2) If $\mathcal{F}\to x(\mathcal{F}^x,*)$, then $\mathcal{G}\to x(\mathcal{F}^x,*)$.

(3) $lim_\tau(\mathcal{F})(\mathcal{F}^x,*)\le lim_\tau(\mathcal{G})(\mathcal{F}^x,*)$.

(4) If $\mathcal{G}\infty x(\mathcal{F}^x,*)$, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$.

(5) $clu_\tau(\mathcal{G})(\mathcal{F}^x,*)\le clu_\tau(\mathcal{F})(\mathcal{F}^x,*)$.

(6) If $\mathcal{F}\to x(\mathcal{F}^x,*)$ and $(L,\odot)$-filter $\mathcal{F}*\mathcal{F}$ exists, then $\mathcal{F}\infty x(\mathcal{F}^x,*)$. In particular, if $\odot = *$ and $\mathcal{F}\to x(\mathcal{F}^x,\odot)$, then $\mathcal{F}\infty x(\mathcal{F}^x,\odot)$ and $lim_\tau(\mathcal{F})(\mathcal{F}^x,\odot)\le clu_\tau(\mathcal{F})(\mathcal{F}^x,\odot)$.

Proof.(1) It follows from Lemma 2.8(3).

(2) It is easily proved from $\mathcal{N}^x_\tau \leq \mathcal{F} \leq \mathcal{G}$ and (1).

(3) From (2), it is clear.

(4) For every $\mathcal{N}^x_\tau(f) * \mathcal{F}(g) \neq \bot$, since $\mathcal{F} \leq \mathcal{G}$,

$$\mathcal{N}^x_\tau(f) * \mathcal{G}(g) \geq \mathcal{N}^x_\tau(f) * \mathcal{F}(g) \neq \bot.$$

Since $\mathcal{G} \infty x(\mathcal{F}^x, *)$, we have $f \odot g \neq 1_\emptyset$. Thus, $\mathcal{F} \infty x(\mathcal{F}^x, *)$.

(5) From (4), it is clear.

(6) Let $\mathcal{F} \to x(\mathcal{F}^x, *)$. For every $\mathcal{N}^x_\tau(f) * \mathcal{F}(g) \neq \bot$, since $\mathcal{N}^x_\tau \leq \mathcal{F}$,

$$(\mathcal{F} * \mathcal{F})(f \odot g) \geq \mathcal{F}(f) * \mathcal{F}(g) \geq \mathcal{N}^x_\tau(f) * \mathcal{F}(g) \neq \bot$$

Hence $(\mathcal{F} * \mathcal{F})(f \odot g) \neq \bot$. It implies $f \odot g \neq 1_\emptyset$. Thus, $\mathcal{F} \infty x(\mathcal{F}^x, *)$. If $\odot = *$, then $\mathcal{F} \odot \mathcal{F} = \mathcal{F}$. Hence the result holds.

Example 4.3 Let $X = \{x, y\}$ be a set, $(L = [0,1], \odot)$ the stsc-quantale with $x \odot y = 0 \vee (x + y - 1)$ and let $f \in [0,1]^X$ defined as $f(x) = 0.6, f(y) = 0.5$. We define a $([0,1], \odot)$-fuzzy topology $\tau : [0,1]^X \to [0,1]$ as follows:

$$\tau(g) = \begin{cases} 1, & \text{if } g \in \{1_X, 1_\emptyset\}, \\ 0.7, & \text{if } g = f, \\ 0.5, & \text{if } g = f \odot f, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Let $\mathcal{F}^x(g) = H^x(g) = [x](g) = g(x)$ be given and $* = \odot$. Then $\mathcal{N}^x_\tau(f) = \bigvee_{g \leq f}(g(x) \odot \tau(g))$. We obtain $([x], \odot)$ and $([y], \odot)$-neighborhood filters $\mathcal{N}^x_\tau, \mathcal{N}^y_\tau : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^x_\tau(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.3, & \text{if } f \leq g \neq 1_X, \\ 0.1, & \text{if } f \odot f \leq g \not\geq f, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{N}^y_\tau(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.2, & \text{if } f \leq g \neq 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}(g) = \inf g$ be given and $g(x) = 0.2, g(y) = 0.05$. Then $\mathcal{G}(g) = \inf g = 0.05 \not\geq \mathcal{N}^x_\tau(g) = 0.1$. Thus, $\mathcal{G} \not\to x([x], \odot)$. Since $\mathcal{G} \geq \mathcal{N}^y_\tau$, $\mathcal{G} \to y([y], \odot)$. Since $h_1 \odot h_2 \geq \mathcal{G}(h_1) \odot \mathcal{N}^z_\tau(h_2) \neq \bot$, for $z \in X$, we have $\mathcal{G} \infty z([z], \odot)$. Thus, $lim_\tau(\mathcal{G})([y], \odot) = \{y\} \subset clu_\tau(\mathcal{G})([x], [y], \odot) = \{x, y\}$.

(2) Let $\mathcal{F}^x(g) = H^x(g) = [x](g) = g(x)$ be given and $* = \wedge$. Then $\mathcal{N}^x_{\tau_i}(f) = \bigvee_{g\le f}(g(x)\wedge\tau(g))$. We obtain $([x],\wedge)$ and $([y],\wedge)$-neighborhood filters $\mathcal{N}^x_\tau, \mathcal{N}^y_\tau : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{N}^x_\tau(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.6, & \text{if } f \le g \ne 1_X, \\ 0.2, & \text{if } f\odot f \le g \not\ge f, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{N}^y_\tau(g) = \begin{cases} 1, & \text{if } g = 1_X, \\ 0.5, & \text{if } f\odot f \le g \ne 1_X, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}(g) = \inf g$ be given. For $\mathcal{G}(f)\wedge\mathcal{N}^x_\tau(f\odot f) = 0.2$, we have $f\odot(f\odot f) = 1_\emptyset$. For $\mathcal{G}(\overline{0.4})\wedge\mathcal{N}^y_\tau(f) = 0.4$, we have $f\odot\overline{0.4} = 1_\emptyset$. Hence $\mathcal{G} \not\propto x([x],\wedge)$ and $\mathcal{G}\not\propto y([y],\wedge)$. Since $\mathcal{G}\ge\mathcal{N}^y_\tau$, then $\mathcal{G}\to y([y],\wedge)$. For $\overline{0.4}\odot\overline{0.3} = 1_\emptyset$, we have $\mathcal{G}(\overline{0.4})\wedge\mathcal{G}(\overline{0.3}) = 0.3$. Hence $\mathcal{G}\wedge\mathcal{G}$ does not exist.

Theorem 4.4 *Let (X,τ) be an $(L,\odot)$-fuzzy topological space, $\mathcal{N}^x_\tau$ be $(\mathcal{F}^x,*)$-neighborhood filter and $\mathcal{F}$ an $(L,\odot)$-filter.*

Then: (1) *If $\mathcal{F}\propto x(\mathcal{F}^x,*)$, then $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$.*

(2) *If $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$ which an $(L,\odot)$-filter $\mathcal{F}*\mathcal{F}$ exists, then $\mathcal{F}\propto x(\mathcal{F}^x,*)$.*

(3) *If $\odot = *$ and $\mathcal{F}$ has a finer $(L,\odot)$-filter $\mathcal{G}$ such that $\mathcal{G}\to x(\mathcal{F}^x,*)$, then $\mathcal{F}\propto x(\mathcal{F}^x,*)$.*

Proof. (1) Since $\mathcal{F}\propto x(\mathcal{F}^x,*)$, for every $\mathcal{N}^x_\tau(f)*\mathcal{F}(g)\ne\bot$, we have $f\odot g\ne 1_\emptyset$. From Theorem 3.1, there exists an $(L,\odot)$- filter $\mathcal{G} = \mathcal{N}^x_\tau * \mathcal{F}$ such that $\mathcal{N}^x_\tau\le\mathcal{G}$ and $\mathcal{F}\le\mathcal{G}$.

(2) Since $\mathcal{N}^x_\tau\le\mathcal{G}$ and $\mathcal{F}\le\mathcal{G}$, for each $\mathcal{N}^x_\tau(f)*\mathcal{F}(g)\ne\bot$,

$$\mathcal{G}(f\odot g)\ge\mathcal{G}(f)*\mathcal{G}(g)\ge\mathcal{N}^x_\tau(f)*\mathcal{F}(g)\ne\bot.$$

Hence $\mathcal{G}(f\odot g)\ne\bot$ implies $f\odot g\ne 1_\emptyset$. It follows $\mathcal{F}\propto x(\mathcal{F}^x,*)$.

(3) Since $\odot = *$, $\mathcal{F}\odot\mathcal{F}$ exists. By (1), it is easily proved.

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Received: November, 2011