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# A Modified Variational Iteration Method for Schrödinger and Laplace Problems

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**Abstract**

In this paper, we apply the modified variational iteration method for solving Schrödinger and Laplace problems which play very important part in applied and engineering sciences. This method choose linear operator for various equations,(linear, nonlinear). So that the Lagrange multiplier can be effectively identified. Several numerical examples are presented to show the ability and efficiency of this method.

**Keywords:** Schrödinger and Laplace problems, variational iteration method

## 1-Introduction

The variational iteration method [1-3] has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti -Sekine-Mura method[4], JI-Huan He [3] developed the variational iteration method in 1999. The variational iteration method ,has been widely applied to solve nonlinear problems, more and more merits have been discovered and some modifications are suggested to overcome the demerits arising in the solution procedure .For example, T.A.Abassy. et al[5,6] also proposed further treatments of these modification results by using pade approximants and the Laplace transform.
Soltani.L.A,Shirzadi.A[7] apply a new modifying of Variational iteration method, which provides great freedom in choosing linear operators for various nonlinear equations. Such technology was also suggested by Ji-Huan He[8-12] for nonlinear oscillators. For example ,consider a nonlinear oscillator

$u'' + u^3 = 0,$

The correction functional can be constructed as follows

$$u_{n+1} = u_n + \int_0^x \lambda(u'' + w^2 u + f),$$

Such treatment is very effective also for non-oscillation equations. We introduce the basic idea underlying the variational iteration method for solving nonlinear equations. Consider the general nonlinear equation:

$$L[u(t)] + N[u(t)] = g(t) \tag{1.1}$$

where $L$ is a linear differential operator, $N$ is a nonlinear operator, and $g$ is a given analytical function. The essence of the method is to construct a correction functional of the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \tag{1.2}$$

where $\lambda$ is a Lagrange multiplier which can be identified optimally via the variational theory M.Inokuti and H.Sekine [4], $u_n$ is the approximate solution and $\tilde{u}_n$ denotes the restricted variation, i.e. $\delta \tilde{u}_n = 0$. After determining the Lagrange multiplier $\lambda$ and selecting an appropriate initial function $u_0$, the successive approximations $u_n$ of the solution $u$ can be readily obtained. Consequently, the solution of Eq. (1.1) is given by $u = \lim_{n\to\infty} u_n$.

In this paper ,we consider the Modified Variational iteration method for Solving Schrödinger and Laplace equation.. Several numerical examples are implemented to show the efficiency of this method.

## 2-A modified variational iteration method

According to the variational iteration method, we consider the following general nonlinear equation:

$$L[u(t)] + N[u(t)] = g(t) \tag{2.1}$$

where $L$ is a linear differential operator, $N$ is a nonlinear operator, and $g$ is a given analytical function .We can construct a correction functional in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \tag{2.2}$$

Where $\tilde{u}_n$ is considered as a restricted variation ,i-e, $\delta \tilde{u}_n = 0$, $\lambda$ is a Lagrange multiplier which can be identified optimally via the variational theory.

For convergence of the sequence obtained via the VIM and its rate, we recall Banach's theorem:

**Theorem .1.( Banach's Fixed point Theorem):**

Assume that $\chi$ is a Banach space,

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$$B : \chi \to \chi$$

is a nonlinear mapping, and suppose that

$$\|B[u] - B[\bar{u}]\| \leq \gamma \|u - \bar{u}\|, \quad \forall u, \bar{u} \in \chi \tag{2.3}$$

for some constant $\gamma < 1$ .Then B has a unique Fixed point .Furthermore, the sequence

$$u_{n+1} = B[u_n] \tag{2.4}$$

with an arbitrary choice of $u_0 \in \chi$ converges to the fixed point of $B$ and

$$\|u_k - u_l\| \leq \|u_1 - u_0\| \sum_{j=l-1}^{k-2} \gamma^j$$

According to the above theorem, for the nonlinear mapping

$$B[u] = u(t) + \int_0^t \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds$$

A sufficient condition for the convergence of the variational iteration method is strictly contraction of $B$ .Furthermore, sequence (2.4) converges to the fixed method of $B$ ,which is also the solution of the equation (2.1). In the above theorem, the rate of convergence depends on $\gamma$ and therefore, in the variation iteration method ,the rate of convergence depends on $\lambda$ . Considering what has been mentioned up until now,a modified variational iteration method can be identified by the following, eq.(2.1) can be rewritten in the following form:

$$L[u(t)] - g_1[u(t)] + g_1[u(t)] + N[u(t)] = g(t) \tag{2.5}$$

where $g_1[u(t)]$ is an arbitrary linear operator of $u(t)$ . Now we can construct a correction functional based on the new linear operator which is:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)(Lu_n(s) - g_1[u_n(s)] + g_1[\tilde{u}_n(s)] + N\tilde{u}_n(s) - g(s))ds \tag{2.6}$$

where $\tilde{u}_n$ is considered as a restricted variation i-e, ,$\delta \tilde{u}_n = 0$ .The Lagrange multiplier , $\lambda$ ,obtained from the correction functional (2.6) is different from (2.2). we can choose the auxiliary linear operator. This provides great freedom in applying the variational iteration method to the(linear,nonlinear) problems.

## 3-Applications

In this section, we give some examples to demonstrate the efficiency and effectiveness of the MVIM. The results are computed by using Maple 13 and compared to exact solution .

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**Example3-1[15]C**onsider the following linear Schrödinger problem:

$$u_t = iu_{xx} \tag{3.1}$$

with the initial condition

$u(x,0) = \sinh(x)$.

which has the exact solution is $u(x,t) = \sinh(x)e^{it}$.

A ccording to the modified variational iteration method, we derive a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left( \frac{\partial u_n}{\partial s} - iu_n + i\tilde{u}_n - i\frac{\partial^2 \tilde{u}_n}{\partial x^2} \right) ds, \tag{3.2}$$

and the stationary condition of the above correction functional can be expressed as:

$$\frac{\partial \lambda(s,t)}{\partial s}\Big|_{s=t} + i\lambda(s,t)\Big|_{s=t} = 0$$

$$1 + \lambda(s,t)\Big|_{s=t} = 0$$

the Lagrange multiplier ,therefore, can be identified as follows:

$$\lambda = -e^{-i(s-t)} \tag{3.3}$$

substituting (3.3) for correction functional (3.2),we have the f ollowing iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{i(t-s)} \lambda \left( \frac{\partial u_n}{\partial s} - i\frac{\partial^2 u_n}{\partial x^2} \right) ds, \tag{3.4}$$

by the variational iteration formula (3.4)and initial approximation ,we get

$u_1(x,t) = \sinh(x)e^{it}$.

in the same way , we obtain,

$u_n(x,t) = \sinh(x)e^{it}$

which means that $u_1(x,t) = u(x,t) = \sinh(x)e^{it}$ is the exact solution.

**Example3-2[17]** Consider the following nonlinear Schrödinger problem:

$$iu_t + u_{xx} - u\cos^2(x) - |u|^2 u = 0, \tag{3.5}$$

with the initial condition

$$u(x,0) = \sin(x)$$

the exact solution is $u(x,t) = \sin(x)e^{-\frac{3}{2}it}$.

According to the modified variational iteration method, we derive a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda \ (i\frac{\partial u_n}{\partial s} - \frac{3}{2}u_n + \frac{3}{2}\tilde{u}_n + \frac{1}{2}\frac{\partial^2 \tilde{u}_n}{\partial x^2} - \tilde{u}_n \cos^2(x) - |\tilde{u}_n|^2 \tilde{u}_n)ds \tag{3.6}$$

and the stationary condition of the above correction functional can be expressed as :

$$\left.\frac{\partial \lambda(s,t)}{\partial s}\right|_{s=t} - \frac{3}{2}i\lambda(s,t)\Big|_{s=t} = 0$$

$$1 + i\lambda(s,t)\big|_{s=t} = 0$$

the Lagrange multiplier ,therefore, can be identified as follows:

$$\lambda = ie^{\frac{3}{2}i(s-t)} \tag{3.7}$$

substituting (3.7) for correction functional (3.6),we have the following iteration formula:

$$u_{n+1} = u_n + \int_0^t ie^{3i(s-t)}(i\frac{\partial u_n}{\partial s} + \frac{1}{2}\frac{\partial^2 u_n}{\partial x^2} - u_n \cos^2(x) - |u_n|^2 u_n)ds \tag{3.8}$$

by the variational iteration formula (3.8), we get

$$u_1(x,t) = \sin(x)e^{-\frac{3}{2}it}$$

which means that $u_1(x,t) = u(x,t) = e^{-\frac{3}{2}it}$, is the exact solution.

**Example3-3[16,18]** Consider the following Laplace problem:

$$u_{xx} + u_{yy} = 0, \quad 0 < x, y < \pi \tag{3.9}$$

with the Dirichlet boundary conditions

$u(x,0)=\sinh(x), \qquad u(x,\pi)=-\sinh(x)$

$u(0,y)=0, \qquad u(\pi,y)=\sinh\pi\cos(y)$

which has the exact solution is $u(x,y)=\sinh(x)\cos(y)$.

According to the modified variational iteration method, we derive a correction functional as follows:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x \lambda \ \left(\frac{\partial^2 u_n}{\partial s^2}-u_n+\tilde{u}_n+\frac{\partial^2 \tilde{u}_n}{\partial y^2}\right)ds \qquad (3.10)$$

and the stationary condition of the above correction functional can be expressed as :

$$\frac{\partial^2 \lambda(s,x)}{\partial s^2}\Big|_{s=x}-\lambda(s,x)\Big|_{s=x}=0$$

$$1-\frac{\partial \lambda(s,x)}{\partial s}\Big|_{s=x}=0$$

$$\lambda(s,x)\Big|_{s=x}=0$$

the Lagrange multiplier ,therefore, can be identified as follows:

$$\lambda=\sinh(s-x) \qquad (3.11)$$

substituting (3.11) for correction functional (3.10),we have the following iteration formula:

$$u_{n+1}=u_n+\int_0^x \sinh(s-x)\left(\frac{\partial^2 u_n}{\partial s^2}+\frac{\partial^2 u_n}{\partial y^2}\right)ds \qquad (3.12)$$

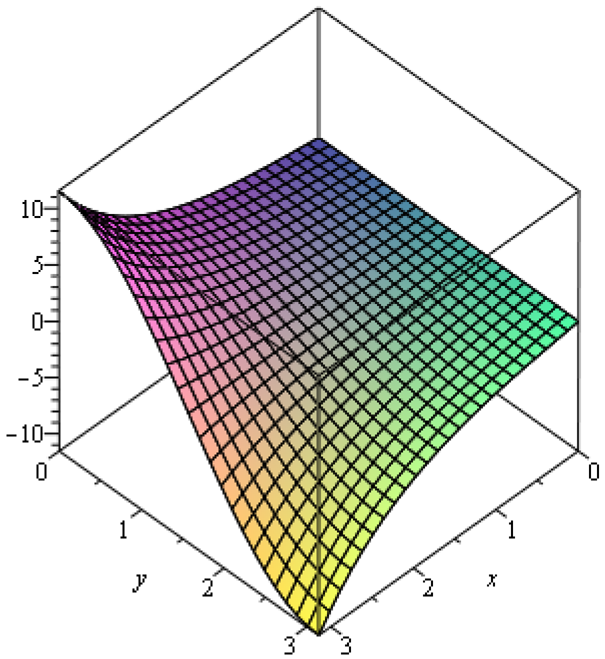
by the variational iteration formula (3.12) and choosing $u_0(0,y)=x\cos(y)$ as initial condition, we get

$u_1(x,y)=\sinh(x)\cos(y)$

in the same way ,we obtain $u_n(x,y)=\sinh(x)\cos(y)$,

which means that $u_1(x,y)=u(x,y)=\sinh(x)\cos(y)$, is the exact solution.

Fig1 The approxiamate solution of MIVM of example(3)




**Example3-4[16, 18]** Consider the following Laplace problem:

$$u_{xx} + u_{yy} = 0, \quad 0 < x, y < \pi \tag{3.13}$$

with the Neumann boundary conditions

$$u_x(0, y) = 0, \qquad u_x(\pi, y) = 0,$$

$$u_y(x,0) = 0, \qquad u_y(x,\pi) = 2\cos(2x)\sinh(2\pi),$$

which has the exact solution is $u(x, y) = \cosh(2y)\cos(2x)$.

According to the modified variational iteration method, we derive a correction functional as follows:

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda \left(\frac{\partial^2 u_n}{\partial s^2} - 4u_n + 4\tilde{u}_n + \frac{\partial^2 \tilde{u}_n}{\partial x^2}\right) ds \tag{3.14}$$

and the stationary condition of the above correction functional can be expressed as :

$$\frac{\partial^2 \lambda(s, y)}{\partial s^2}\Big|_{s=y} - 4\lambda(s, y)\Big|_{s=y} = 0$$

$$1-\frac{\partial\lambda(s,y)}{\partial s}\bigg|_{s=y}=0$$

$$\lambda(s,y)\big|_{s=y}=0$$

the Lagrange multiplier ,therefore, can be identified as follows:

$$\lambda=\frac{1}{2}\sinh 2(s-y) \tag{3.15}$$

substituting (3.15) for correction functional (3.14), we have the following iteration formula:

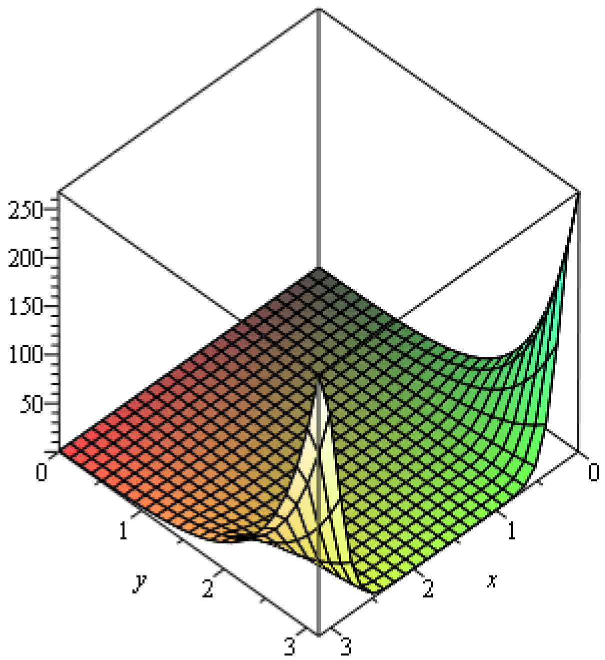
$$u_{n+1}(x,y)=u_n(x,y)+\int_0^y \frac{1}{2}\sinh 2(s-y)\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial s^2}\right)ds \tag{3.16}$$

by the variational iteration formula (3.16) and choosing $u_0(x,0)=(1+2y^2)\cos(2x)$ as initial condition, we get:

$$u_1(x,y)=\cosh(2y)\cos(2x)$$

in the same way, we obtain $u_n(x,y)=\cosh(2y)\cos(2x)$
which means that $u_1(x,y)=u(x,y)=\cosh(2y)\cos(2x)$, is the exact solution.

Fig.2 The approxiamate solution of MIVM of example(4)



## 5-Conclusion

In the work ,we proposed a modified variational iteration method to solve Schrödinger and Laplace Problems ,we see a modified variational method shows the convergence is fast from the VIM to the closed from solution. The obtained solution shows that MVIM is a very convenient and effective for Schrödinger *and* Laplace Problems , only one iteration leads to exact solutions.

## References

[1] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media,Comput. Methods Appl. Mech. Engrg. 167(1998)57-68.
[2] J. H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg. 167(1998)69-73.
[3] J.-H. He, Variational iteration method _ a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear.Mech. 34 (1999) 699_708.
[4] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in non-linear mathematical physics, Variational Methods in the Mechanics of Solids, Pergamon Press, New York, 1978, pp. 156–162.
[5] T. A. Abassy , M.A.El-Tawil ,H.El-Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration Pade technique ,Journal of Computational and Applied Mathematics 207 (1)(2007)73-91.
[6] T. A. Abassy, M.A.El-Tawil ,H.El-Zoheiry, Exact solutions of some nonlinear partial differential using the variational iteration method linked with Laplace transforms and the Pade technique, Computers and Mathematics with Applications,54 (7-8))(2007)940-954.
[7] L. A. soltani,A.shirzadi,Anew modification of the variational iteration method,Computers and Mathematics With Applications 59(2010)2528-2535.
[8] J.Huan,He,Zhong-FuRen, Asimple approach to nonlinear oscillators, Physics LittersA 373(2009)3749-3752.
[9] J.Huan. He, Preliminary report on the energy balance for nonlinear oscillations, Mechanics Research Communications 29(200)107-111.
[10] J.Huan. He, The homotopy perturbation method for nonlinear oscillators, with discontinuities ,Applied Mathematics and Computation 151(2004)287-292.
[11] J.Huan. He, Variational approach for nonlinear oscillators, Chaos,Solitons and Fractals 34(2007)1430-1439.
[12] J. Huan. He, An improved amplitude-frequency for nonlinear oscillators, International Joumal of Nonlinear sciences and Numerical simulation 9(2008)211.
[13] J.H.He, Xu-Hong, Variational iteration method :New development and applications, computers and Mathematics with Applications 54(2007) 881-894.

[14] J.H.He,G.C.Wu,F.Austin,Variational iteration method which should be followed, Nonlinear Science Letters A1(2010)1-30.

[15] A. M.Wazwaz, Partial Differential Equation;Methods and Applications,Bel-kema publishers, The Netherlands,(2002).

[16] A. M.Wazwaz,The variation iteration method for exact solutions of Laplace equation ,Phys,Lett.A,363,260-262,(2007) .

[17] A. Borhanifar, Reza Abazari, Numerical study of nonlinear Schrödinger and coupled Schrödinger equations by differential transformation method, Optics Communications 283 (2010) 2026–2031.

[18] M. A. Noor, Modified Variational Iteration Method for Goursat and Laplace Problems, World Applied Sciences Journal 4 (4): 487-498, 2008.

[19] A. Neamaty and R. Darzi, comparison between the Variational iteration method and the Homotopy Perturbation method for the Sturm-Liouville differential Equation,Hindawi publishing Corporation, Boundary Value Problems ,Volume 2010,Article ID317369,14 pages ,doi:10.1155/2010/317369.

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