Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 14, 685 - 692

On Solutions of the Equations $x^2 \pm y^8 = \pm z^6$

[1]Selin (İNAĞ) ÇENBERCİ, [2]Bilge PEKER and [3]Sema COŞKUN

[1,3]Mathematics Teaching Programme Graduate School of
Natural and Applied Sciences
Ahmet Kelesoglu Education Faculty
Selcuk University, Konya, Turkey
inag_s@hotmail.com, semacoskun@selcuk.edu.tr

[2]Elementary Mathematics Education Programme
Ahmet Kelesoglu Education Faculty
Selcuk University, Konya, Turkey
bilge.peker@yahoo.com

Abstract

In this article, we determine all solutions to the equation $x^a + y^b = z^c$, $(a, b, c) \in \{(2, 8, 6), (2, 6, 8), (8, 6, 2)\}$ in coprime integers x, y, z. In this case we get a set of curves by doing some transformations to these equations. Then we determine the rational points and some features of these curves.

1 Introduction

Fermat's Last Theorem is the famous problem of Number Theory. This theorem states that if $n > 2$, then the equation

$$a^n + b^n = c^n$$

has no solutions in nonzero integers a, b, c. If we let $x = \dfrac{a}{c}$ and $y = \dfrac{b}{c}$, then solutions of Fermat's equation give rational points on the Fermat's curve

$$x^p + y^p = 1.$$

But Fermat's curve is not an elliptic curve.

Gerhard Frey and others suggested using on hypothetical solution (a, b, c) of Fermat's equation to manufacture an elliptic curve

$$F_{a,b,c} = y^2 = x(x - a^p)(x + b^p)$$

and A. Wiles completed the proof of Fermat's Last Theorem with the aid of elliptic curves. The first instances of elliptic curves occur in the works of Diophantus and Fermat.

An elliptic curve is a curve given by an equation

$$E : y^2 = f(x)$$

for a cubic or quartic polynomial of x [9].

Elliptic curves are plane curves that are the locus of points satisfying a cubic equation in two variables. If the elliptic curve is defined on Euclidean plane, the points which are related to this elliptic curve will be affine rational points. And if the elliptic curve is defined over $\mathbb{Q}$, the points which are related to this elliptic curve will be rational points.

After a short introduction to elliptic curves, now we mention about the Diophantine equation belongs to our article.

In this article, we consider the general Diophantine equation $x^p + y^q = z^r$. This Diophantine equation in integers $p > 1,\ q > 1,\ r > 1$ and x, y, z is a generalization of the well-known Fermat equation $x^n + y^n = z^n$.

There are a vast amount of results related to the equation $Ax^p + By^q = Cz^r$. The quantity $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ determines the general shape of the set of primitive solutions. One of the most important result for this equation is a theorem by Darmon and Granville [5] which for fixed, non-zero A, B, C and for fixed p, q, r satisfying the quantity $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ relates primitive solutions $x, y, z \in \mathbb{Z}$ and $\gcd(x, y, z) = 1$. Their theorem implies that when $\chi < 1$ and $A, B, C \in \mathbb{Z}$ with $ABC \neq 0$ the number of solutions with $\gcd(x, y, z) = 1$ to the equation $Ax^p + By^q = Cz^r$ is finite.

A special case of interest is when $A = B = C = 1$. In many such cases the solution set has been found. Below it can be listed the exponent triples (p, q, r) of solved equations together with the non-trivial solutions $(xyz \neq 0)$.

Bruin [1] determined all solutions for the case $(p, q, r) \in \{(2, 4, 6), (2, 6, 4), (4, 6, 2), (2, 8, 3)\}$ in coprime integers x, y, z.

Poonen [8] researched the solutions of the equation for $(p, q, r) \in \{(5, 5, 2), (9, 9, 2), (5, 5, 3)\}$.

Bruin [2] clarified the rational integers x, y, z such that $(p, q, r) = (3, 9, 2)$ and $\gcd(x, y, z) = 1$.

In the light of these studies we consider the equation $x^2 \pm y^8 = \pm z^6$. In order to calculate some values related to these equations, we use GP/Pari [6].

GP/Pari is widely used computer algebra system designed for fast computations in number theory. This programme is originally (1985-1996) developed at Universite Bordeaux I by a team led by Henri Cohen.

2 Preliminaries

For solving the equation, we shall use the parametrizations of some equations. Before our theorems we give two lemmas for proof of theorems.

Lemma 1 [7] *Let x, y, z be coprime integers such that $x^2+y^2 = z^2$. Possibly by interchanging x and y, we can assume that x is divisible by 2. Then there are coprime integers s and t, not both odd such that*

$$x = 2st \qquad \pm y = s^2 - t^2 \qquad \pm z = s^2 + t^2.$$

Lemma 2 [3] *Let x, y, z be coprime integers such that $x^2 + y^2 = z^3$. Then there are coprime integers s, t such that*

$$x = s(s^2 - 3t^2) \qquad y = t(t^2 - 3s^2) \qquad z = s^2 + t^2.$$

3 The Equations $x^2 = z^6 + y^8$, $x^2 + y^8 = z^6$ and $x^2 + z^6 = y^8$

And now we can give our first theorem.

Theorem 3 *If $x, y, z \in \mathbb{Z}$ are coprime such that $x^2 = z^6 + y^8$, then $xyz = 0$.*

Proof. Suppose we have a primitive solution x, y, z. Then *Lemma* 3 states that there exists coprime integers s, t of distinct parity such that $y^4 = 2st$, $z^3 = s^2 - t^2$ or $y^4 = s^2 - t^2$, $z^3 = 2st$. We treat these cases separately.

The case $y^4 = 2st$ and $z^3 =$ $s^2 - t^2 = (s-t)(s+t)$

Since $\gcd(x, y) = 1$ and $(s - t)$, $(s + t)$ are both odd , then we have $\gcd((s - t), (s + t)) = 1$. Therefore there exist $u, v \in \mathbb{Z}$ such that $u^3 = s - t$, $v^3 = s + t$. From these $s = \dfrac{u^3 + v^3}{2}$, $t = \dfrac{u^3 - v^3}{2}$. If we rewrite y^4 in u, v, then it gives

$$y^4 = 2st$$

$$y^4 = 2\frac{u^3+v^3}{2}\frac{u^3-v^3}{2}$$

$$2y^4 = u^6 - v^6$$

$u = 0$ implies that $s = t$ and thus $z = 0$.

In the last equation, we can therefore safely put $Z = \frac{u^2}{v^2}$, $Y = \frac{y^2}{v^3}$ solutions with $z \neq 0$.

$$2Y^2v^6 = (Z^3v^6 - v^6)$$

$$2Y^2 = Z^3 - 1$$

If we change $Y = \frac{y}{\sqrt{2}}$, then we obtain other solutions correspond to the affine rational points on the elliptic curve below.

$$2\frac{y^2}{2} = Z^3 - 1$$

$$y^2 = Z^3 - 1.$$

For this elliptic curve, we can calculate following values by using GP/Pari.

j-invariant	$\triangle$-discriminant	N-conductor	$v = [u, r, s, t]$
0	-432	144	$v = [1, 0, 0, 0]$

From the table, the value $v = [1, 0, 0, 0]$ means that this elliptic curve is minimal. This curve has only one affine rational point namely $(Z, y) = (1, 0)$. This point corresponds to the solution with $y = 0$.

The case $y^4 = s^2 - t^2$ and $z^3 = 2st$

Since y is odd, we have $y^4 \equiv 1(Mod\ 4)$. Therefore s is odd. From $z^3 = 2st$ then we conclude that $s = v^3$ and $t = 4u^3$. If we rewrite y^4 in u, v, then it gives us

$$y^4 = s^2 - t^2$$

$$y^4 = v^6 - 16u^6.$$

Note that $u = 0$ implies that $t = 0$ and thus $z = 0$. We can safely put $Z = \frac{v^2}{u^2}$, $Y = \frac{y^2}{u^3}$, then we get other solutions with $z \neq 0$, correspond to affine rational points on the elliptic curve below.

$$y^4 = v^6 - 16u^6$$

$$Y^2u^6 = Z^3u^6 - 16u^6$$

$$Y^2 = Z^3 - 16$$

This elliptic curve has the values showed in the following table.

j-invariant	$\triangle$-discriminant	N-conductor	$v = [u, r, s, t]$
0	-110592	432	$v = [1, 0, 0, 0]$

Hence we get a minimal polynomial and the curve has no affine rational points. Consequently, for both of the cases we get one variable of our main equation equals to 0.

This completes the proof. ∎

Theorem 4 *If* $x, y, z \in \mathbb{Z}$ *are coprime such that* $x^2 + y^8 = z^6$, *then* $xyz = 0$.

Proof. Observe that any primitive solution of the Diophantine equation $x^2 + y^8 = z^6$ must also satisfy $x^2 + (y^4)^2 = (z^2)^3$. From *Lemma* 4, there exists coprime integers s, t such that $x = t(3s^2 - t^2)$, $y^4 = s(s^2 - 3t^2)$ and $z^2 = s^2 + t^2$. Since $z^2 = s^2 + t^2$, we gain s, t values from Pythagorean triples either $s = a^2 - b^2$, $t = 2ab$ or $s = 2ab$, $t = a^2 - b^2$. We treat each of the cases separately.

The case $s = a^2 - b^2$, $t = 2ab$

In this case, if we rewrite y^4 in a, b, then we get the equation below.

$$y^4 = s(s^2 - 3t^2)$$

$$y^4 = (a^2 - b^2)[(a^2 - b^2)^2 - 3(2ab)^2]$$

$$y^4 = (a^2 - b^2)(a^4 + b^4 - 8a^2b^2)$$

Note that $b = 0$ implies that $t = 0$ and thus $x = 0$. Therefore we can safely put $Y = \frac{y^2}{b^3}$ and $X = \frac{a}{b}$. Solutions with $x \neq 0$ correspond to affine rational points on the elliptic curve.

$$Y^2b^6 = (X^2b^2 - b^2)(X^4b^4 + b^4 - 8X^2b^4)$$

$$Y^2 = (X^2 - 1)(X^4 - 8X^2 + 1)$$

By putting $X^2 = k$, we get

$$Y^2 = k^3 - 9k^2 + 9k - 1.$$

For this elliptic curve we can give following values by using GP/Pari.

j-invariant	$\triangle$-discriminant	N-conductor	$v = [u, r, s, t]$
$\frac{93312}{5}$	34560	5760	$v = [1, 3, 0, 0]$

For this elliptic curve there is only one affine rational point, $(k, Y) = (1, 0)$. Hence we get $(X, Y) \in \{(1, 0), (-1, 0)\}$. These points correspond to the solution with $y = 0$.

The case $s = 2ab$, $t = a^2 - b^2$

For this case, we can put $a - b = u$ and $a + b = v$. This gives $t = uv$. Since $a = \frac{u+v}{2}$, $b = \frac{u-v}{2}$, we obtain $s = \frac{v^2 - u^2}{2}$. By rewriting y^4 in u,v, the below equation is obtained

$$y^4 = s(s^2 - 3t^2)$$

$$y^4 = \frac{v^2 - u^2}{2}((\frac{v^2 - u^2}{2})^2 - 3(uv)^2)$$

$$8y^4 = (v^2 - u^2)(v^4 + u^4 - 14u^2v^2).$$

Note that $u = 0$ implies that $b = a$ thus $x = 0$. By putting $Y = \frac{y^2}{u^3}$, $X = \frac{v^2}{u^2}$, we obtain

$$8Y^2u^6 = (X^2u^2 - u^2)(X^2u^4 - 14Xu^4 + u^4)$$

$$8Y^2 = (X^2 - 1)(X^2 - 14X + 1).$$

This curve is isomorphic to 576A2 in [4] has only one affine rational point, namely $(1, 0)$. This corresponds to solutions with $y = 0$.

This completes the proof. ■

Theorem 5 *If $x, y, z \in \mathbb{Z}$ are coprime such that $x^2 + z^6 = y^8$, then $xyz = 0$.*

Proof. Suppose we have a primitive solution x, y, z. If $z \neq 0$, then $y^8 - x^2 > 0$. Therefore both $y^4 - x$ and $y^4 + x$ are greater than 0. Since x, y are coprime, $\gcd((y^4 - x)(y^4 + x)) \mid 2$. So we get $y^4 - x = 2u^6,\ y^4 + x = 2^5 v^6$ or $y^4 - x = u^6,\ y^4 + x = v^6$. We treat these cases separately.

The case $y^4 - x = 2u^6$, $y^4 + x = 2^5 v^6$

By eliminating x from the equations, we have

$$2y^4 = 2u^6 + 2^5 v^6$$

$$y^4 = u^6 + 16v^6.$$

$v = 0$ implies that $z = 0$. So we must take $v \neq 0$. Therefore we safely put $Y = \frac{y^2}{v^3}$, $X = \frac{u^2}{v^2}$, then we get

$$y^4 = u^6 + 16v^6$$

$$Y^2 v^6 = X^3 v^6 + 16v^6$$

$$Y^2 = X^3 + 16.$$

This curve has values like that the following table

j-invariant	$\triangle$-discriminant	N-conductor	$v = [u, r, s, t]$
0	-110592	27	$v = [2, 0, 0, 4]$

Other solutions correspond to affine rational points on the elliptic curve $Y^2 = X^3 + 16$ which is isomorphic to 27A3 in [4] has only two affine rational points $(0, -4)$ and $(0, 4)$. The corresponding solutions have $u = z = 0$.

The case $y^4 - x = u^6,\ y^4 + x = v^6$

In this case, we get $2Y^2 = X^3 + 1$. In [1], Bruin gave 8 affine rational points: $\{\infty^+, \infty^-, (0, \pm 1), (\pm 1, \pm 1)\}$ for this curve. Solutions corresponding to these points implies that $x = 0$.

This completes the proof. ■

References

[1] N. Bruin, The Diophantine equations $x^2 \pm y^4 = \pm z^6$ and $x^2 + y^8 = z^3$, Compositio Math. 118 (1999), 305-321.

[2] N. Bruin, The primitive Solutions to $x^3 + y^9 = z^2$, Journal of Number Theory 111 (2005), 179-189.

[3] H. Cohen, Number Theory Volume I : Tools and Diophantine Equations, Springer Science+Business Media, New York, 2007.

[4] J.E. Cremona, Algorithms for modular Elliptic Curves, Cambridge University Press, 1992.

[5] H. Darmon, A. Granville, On the equations $z^m = F(x,y)$ and $Ax^p + By^q = Cz^r$, Bull. London Math. Soc. 27(6) (1995), 513-543.

[6] GP/PARI, http://www.math.u-bordeaux.fr/∼belabas/pari/doc/fag.html

[7] L.J. Mordell, Diophantine equations, Academic Press, London, 1969.

[8] B. Poonen, Some Diophantine equations of the form $x^n + y^n = z^m$, Acta Arith. LXXXVI.3 (1998), 193–205.

[9] J.H. Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag, New York, 1986.

Received: October, 2011