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The Generalized Order-k Jacobsthal Numbers

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Abstract

In this paper, we consider the usual Jacobsthal numbers and defined a new sequence which is called generalized order-k Jacobsthal sequence. Then, we investigate some properties of the sequence, obtain generalized Binet formula and give a formula for sums of the Jacobsthal numbers.

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1 Introduction

It is known that the Jacobsthal sequence is defined by the following equation, for $n \geq 2$

$$J_n = J_{n-1} + 2J_{n-2},$$

where $J_0 = 0$ and $J_1 = 1$.

The Jacobsthal sequence is a special case of a sequence which is defined as a linear combination by Kalman, as following

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n,$$

where $c_1, c_2, ..., c_k$ are real constants. Kalman [1] showed that number sequences can be derived by a matrix representation. He derived closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} c_1 & c_2 & ... & c_{k-1} & c_k \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}.$$

Then by an inductive argument, he obtained

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix},$$

where a_n is the n^{th} term of the sequence.

In [3, 4] authors investigated some properties involving Jacobsthal numbers. Also in [4], authors gave matrix method for generating Jacobsthal sequence as following:

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

By taking positive integer powers of this matrix, it can be easily obtain that

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

In [5], Tascı and Kılıc defined order-k Lucas sequence in matrix representation by employing the matrix methods of Kalman. Also, authors give generalized Binet formula and sums of the generalized order-k Pell numbers [2].

At the present paper, we give a new generalization of the Jacobsthal numbers in matrix representation and obtain some properties of the sequence by matrix methods.

2 The Main Resuts

Define k-sequences of the generalized order-k Jacobsthal numbers as shown:

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + J_{n-3}^i + \cdots + J_{n-k}^i, \tag{1}$$

for $n > 0$ and $1 \le i \le k$, with initial conditions

$$J_n^i = \begin{cases} 1 & \text{if } i+n=1, \\ 0 & \text{otherwise,} \end{cases} \text{ for } 1-k \le n \le 0,$$

where J_n^i is the n^{th} term of the i^{th} sequence. For $k = 2$ and $i = 1$ the generalized order-k Jacobsthal sequence is reduced to the the conventional Jacobsthal sequence.

By the definition of generalized Jacobsthal numbers, we can write following vector recurrence relation

$$\begin{bmatrix} J_{n+1}^i \\ J_n^i \\ J_{n-1}^i \\ \vdots \\ J_{n-k+2}^i \end{bmatrix} = C \begin{bmatrix} J_n^i \\ J_{n-1}^i \\ J_{n-2}^i \\ \vdots \\ J_{n-k+1}^i \end{bmatrix}, \tag{2}$$

where C is a k-square companion matrix as following:

$$C = \begin{bmatrix} 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3}$$

The matrix C is called to be generalized order-k Jacobsthal matrix. Let us to define a $k-$square matrix $B_n = [b_{ij}]$ to deal with the k sequences of the generalized order-k Jacobsthal numbers, as following:

$$B_n = \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ J_{n-k+1}^1 & J_{n-k+1}^2 & \cdots & J_{n-k+1}^k \end{bmatrix} \tag{4}$$

If we expand (2) to k columns, we obtain the following matrix equation:

$$B_n = CB_{n-1}. \tag{5}$$

Then we have the following lemma.

Lemma 1 *Let C and B_n be as in (3) and (4), respectively. Then for all integers $n \geq 0$*

$$B_n = C^n.$$

Proof. By (4), we have $B_n = CB_{n-1}$. Then, by an inductive argument, we can write it as

$$B_n = C^{n-1}B_1.$$

By definition of the generalized order-k Jacobsthal numbers, $B_1 = C$; therefore

$$B_n = C^n,$$

which is desired. ■

Corollary 2 *Let B_n be as in (4). Then*

$$\det B_n = \begin{cases} -2, & \text{if } k = 2 \\ 1, & \text{if } k \text{ is odd} \\ -1, & \text{if } k \text{ is even, } (k \neq 2) \end{cases}.$$

Proof. From Lemma 1, we know $B_n = C^n$. Then

$$\det B_n = \det C^n = \det(C)^n.$$

By the Laplace expansion of determinant with respect to the any column, it is easy to compute the determinant of C. So the proof is complete. ■

Now we give some relations involving the generalized order-k Jacobsthal numbers.

Lemma 3 *Let J_n^i be the n^{th} generalized order-k Jacobsthal number. Then*

$$\begin{aligned} J_{n+1}^1 &= J_n^1 + J_n^2 \\ J_{n+1}^2 &= 2J_n^1 + J_n^3 \\ J_{n+1}^i &= J_n^1 + J_n^{i+1};\ 3 \leq i \leq k-1 \\ J_n^1 &= J_{n+1}^k. \end{aligned}$$

Proof. We know from (4), $B_{n+1} = B_n B_1$. By using the matrix multiplication the proof is readily seen. ■

Some generalized order-k Jacobsthal numbers are given in Table 1.

	$k=$	2	k	$=$	3		k	$=$	4
$n\backslash i$	1	2	1	2	3	1	2	3	4
-2	0	0	0	0	1	0	0	1	0
-1	0	1	0	1	0	0	1	0	0
0	1	0	1	0	0	1	0	0	0
1	1	2	1	2	1	1	2	1	1
2	3	2	3	3	1	3	3	2	1
3	5	6	6	7	3	6	8	4	3
4	11	10	13	15	6	14	16	9	6
5	21	22	28	32	13	30	37	20	14

Table1

3 Generalized Binet Formula

In this section we derived a generalized Binet formula for generalized order-k Jacobsthal numbers. From companion matrices, it is known that the chacteristic equation of the matrix C is $x^k - x^{k-1} - 2x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$, which is also the characteristic equation of generalized order-k Jacobsthal numbers.

Lemma 4 *The equation $x^{k+1} - 2x^k - x^{k-1} + x^{k-2} + 1$ does not have multiple roots for $k \geq 3$.*
Proof. *Let $f(x) = x^k - x^{k-1} - 2x^{k-2} - x^{k-3} - \cdots - x - 1$ and*

$$g(x) = (x-1)f(x) = x^{k+1} - 2x^k - x^{k-1} + x^{k-2} + 1.$$

It is easy to see that, 1 is a root of $g(x)$ but not a multiple root, since $k \geq 3$ and $f(1) \neq 0$. Suppose that β is a multiple root of $g(x)$ such that $\beta \neq 0$ and $\beta \neq 1$. Since β is a multiple root

$$\begin{aligned} g(\beta) &= \beta^{k+1} - 2\beta^k - \beta^{k-1} + \beta^{k-2} + 1 \\ &= \beta^{k-2}[\beta^3 - 2\beta^2 - \beta + 1] + 1 = 0 \end{aligned}$$

and

$$\begin{aligned} g'(\beta) &= (k+1)\beta^k - 2k\beta^{k-1} - (k-1)\beta^{k-2} + (k-2)\beta^{k-3} = 0 \\ &= \beta^{k-3}[(k+1)\beta^3 - 2k\beta^2 - (k-1)\beta + k + 2] = 0. \end{aligned}$$

Thus

$$\beta_1 = \frac{1}{3(k+1)}\left[\frac{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}}{2} + \frac{14k^2-6}{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}} + 2k\right] \text{ and}$$

$$\begin{aligned}\beta_{2,3} \;=\; & -\frac{1}{3(k+1)}\left\{\frac{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}}{4} - \frac{7k^2-3}{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}} + 2k\right. \\ & \left.\pm\frac{\sqrt{3}}{2}i\left[\frac{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}}{2} - \frac{14k^2-6}{[a+12\sqrt{b}k+\sqrt{b}]^{1/3}}\right]\right\},\end{aligned}$$

where $a = 28k^3 - 612k - 432k^2 - 216$ *and* $b = -147k^4 + 126k^3 + 1149k^2 + 1164k + 336$. *It is easy to see that* β_i*'s are distinct from each other. Hence*

$$\begin{aligned} 0 &= -g(\beta_i) = \beta_i^{k-2}[-\beta_i^3 + 2\beta_i^2 + \beta_i - 1] - 1 \qquad (6)\\ &= u_{k,i} - 1,\end{aligned}$$

where $u_{k,i} = \beta_i^{k-2}[-\beta_i^3 + 2\beta_i^2 + \beta_i - 1]$. *By choosing* $k = 3$ *and* $1 \le i \le 3$, *(6) can be written as*

$$\begin{aligned} 0 &= -g(\beta_1) = \beta_1[-\beta_1^3 + 2\beta_1^2 + \beta_1 - 1] - 1\\ &= u_{3,1} - 1 = 0.\end{aligned}$$

and $u_{3,1} = -0.8445476618 \neq 1$ *is a contradiction. Similarly for* β_2

$$0 = -g(\beta_2) = \beta_2[-\beta_2^3 + 2\beta_2^2 + \beta_2 - 1] - 1$$

and $u_{3,2} = \beta_2[-\beta_2^3 + 2\beta_2^2 + \beta_2 - 1] = 1.172273831 + 0.3253556687i \neq 1$ *is a contradiction and by similar way for* β_3, $u_{3,3} = \beta_3[-\beta_3^3 + 2\beta_3^2 + \beta_3 - 1] = 1.172273831 - 0.3253556687i \neq 1$. *This is also a contradiction because we did suppose that* β *is a multiple root for any integers* $k \geq 3$. *Therefore, the equation* $g(x) = 0$ *does not have multiple roots.* ■

Consequently, from Lemma 4, the equation $x^k - x^{k-1} - 2x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$ does not have multiple roots for $k \geq 3$.

Let $f(\lambda)$ be the characteristic polynomial of the generalized order-k Jacobsthal matrix C. Then, by Lemma 4, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of matrix C. Let V be Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \cdots & \lambda_1 & 1\\ \lambda_2^{k-1} & \lambda_2^{k-2} & \cdots & \lambda_2 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \lambda_k^{k-1} & \lambda_k^{k-2} & \cdots & \lambda_k & 1\end{bmatrix}.$$

Let c_k^i be a $k \times 1$ matrix

$$c_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $V_j^{(i)}$ be $k-$square matrix obtained from V by replacing the j^{th} column of V by c_k^i.Then we obtain the generalized Binet formula for the generalized order-k Jacobsthal numbers by the following theorem.

Theorem 5 *Let J_n^i be the n^{th} term of i^{th} Jacobsthal sequence, for $1 \leq i \leq k$. Then*

$$J_{n-i+1}^j = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Proof. C is diagonizable, due to its eigenvalues are distinct. Denote $V^T = D$ and D is invertible. Then we can write

$$D^{-1}CD = diag(\lambda_1, \lambda_2, \ldots, \lambda_k) = \Lambda$$

Hence C is similar to Λ. So we obtain $C^n D = D\Lambda^n$. It is known that $B_n = C^n$ from Lemma 1. Then we have the following linear system of equations:

$$\begin{aligned} b_{i1}\lambda_1^{k-1} + b_{i2}\lambda_1^{k-2} + \cdots + b_{ik} &= \lambda_1^{n+k-i} \\ b_{i1}\lambda_2^{k-1} + b_{i2}\lambda_2^{k-2} + \cdots + b_{ik} &= \lambda_2^{n+k-i} \\ &\vdots \\ b_{i1}\lambda_k^{k-1} + b_{i2}\lambda_k^{k-2} + \cdots + b_{ik} &= \lambda_k^{n+k-i}, \end{aligned}$$

where $B_n = [b_{ij}]_{k\times k}$. Thus we obtain for each $j = 1, 2, ..., k$,

$$b_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

i.e., $b_{ij} = J_{n-i+1}^j$. So we have the conclusion. ■

4 Sums of the Jacobsthal numbers

In this section, we extend the matrix representation and obtain the sums of the generalized Jacobsthal numbers. The sums T_n of the generalized order-k Jacobsthal numbers are defined by

$$T_n = \sum_{i=0}^{n-1} J_i^1.$$

Since $J_n^1 = J_{n+1}^k$ which is given in Lemma 4, we can rewrite it as

$$T_n = \sum_{i=1}^{n} J_i^k.$$

Let E and W_n be $(k+1)$-square matrices such that

$$E = \begin{bmatrix} 1 & 0 \;\; 0 \;\; \cdots \;\; 0 \\ 1 & \\ 0 & \\ \vdots & C \\ 0 & \end{bmatrix}$$

and

$$W_n = \begin{bmatrix} 1 & 0 \;\; 0 \;\; \cdots \;\; 0 \\ T_n & \\ T_{n-1} & B_n \\ \vdots & \\ T_{n-k+1} & \end{bmatrix},$$

where C and B_n are k-square matrices as in (3) and (4), respectively. Using the formula $J_n^1 = J_{n+1}^k$, we obtain

$$T_n = J_{n-1}^1 + T_{n-1}$$

so we derive the following matrix recurrence equation:

$$W_n = W_{n-1}E.$$

By more generalization, we also have

$$W_n = W_1 E^{n-1}.$$

Since $T_{-i} = 0;\ 1 \leq i \leq k$ and by the definition of the generalized order-k Jacobsthal numbers, we get $W_1 = E$, and in general, $W_n = E^n$. So we obtain the generating matrix for the sums of the generalized order-k Jacobsthal numbers. Since $W_n = E^n$, we may write

$$W_{n+1} = W_n W_1 = W_1 W_n \tag{7}$$

which shows that W_1 is commutative with W_n as well matrix multiplication. By an application of equation (7), the sums of the generalized order-k Jacobsthal numbers satisfy the recurrence relation :

$$T_n = 1 + T_{n-1} + 2T_{n-2} + \sum_{i=3}^{k} T_{n-i}. \tag{8}$$

For example, when $k = 2$ and $i = 1$ the sequence is reduced to the usual Jacobsthal sequence. So the equation (8) becomes

$$T_n = \sum_{i=0}^{n-1} J_i^1 = 1 + T_{n-1} + 2T_{n-2}.$$

Since $T_n = J_{n-1}^1 + T_{n-1}$, the sums of the Jacobsthal numbers are

$$\sum_{i=0}^{n-1} J_i = \frac{3J_{n-1} + 2J_{n-2} - 1}{2}.$$

Corollary 6 *Let J_i be the i^{th} Jacobsthal number. Then,*

$$\sum_{i=0}^{n-1} J_i = \begin{cases} \frac{3J_{n-1}+2J_{n-2}-1}{2} = J_n, & \text{if } n \text{ is even} \\ \frac{3J_{n-1}+2J_{n-2}-1}{2} = J_n + 1, & \text{if } n \text{ is odd} \end{cases}.$$

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