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On the Number of Minimal Dominating Sets Including the Set of Leaves in Trees

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Abstract

A subset $Q \subseteq V(G)$ is a dominating set of a graph G if each vertex in $V(G)$ is either in Q or is adjacent to a vertex in Q. A minimal dominating set Q of a graph G is a dominating set that contains no dominating set of G as a proper subset. In this paper we study the number of all minimal dominating sets in trees including the set of leaves. We determine the smallest number and the largest number of minimal dominating sets including the set of leaves among n-vertex trees. Corresponding trees with those numbers are characterized.

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1 Introduction

In general we use the standard terminology and notation of graph theory, see [2]. Only simple undirected graphs are considered. By P_n, $n \geq 2$, we mean graph with the vertex set $V(P_n) = \{x_1, \ldots, x_n\}$ and the edge set $E(P_n) = \{\{x_i, x_{i+1}\}; i = 1, \ldots, n-1\}$. Moreover, P_1 is a graph with one vertex and P_0 is a graph with $V(P_0) = \emptyset$. By the *subdivision of an edge* $e = \{x, y\}$ of G we mean inserting a new vertex of degree 2 into the edge e. We denote it by $sub_{\{x,y\}}(G)$. By *p-th subdivision,* $p \geq 1$, *of the star* $K_{1,n-1}$ we mean a graph obtained by subdivision of p arbitrary edges of $K_{1,n-1}$ and we denote it by $sub_p(K_{1,n-1})$. If $\{x, y\} \in E(G)$ then we say that x is a *neighbor* of y. The set of all neighbors of x is called the *open neighborhood* of x and is

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denoted by $N(x)$. The *closed neighborhood* of x is $N[x] = N(x) \cup \{x\}$. For a subset $X \subseteq V(G)$ we put $N(X)$ and $N[X]$ instead of $\bigcup_{x \in X} N(x)$ and $\bigcup_{x \in X} N[x]$, respectively. Let $X \subset V(G) \cup E(G)$. The notation $G \setminus X$ means the graph obtained from G by deleting the set X. A subset $Q \subseteq V(G)$ is a *dominating set* of G if for every vertex $x \in V(G)$, $|N[x] \cap Q| \geq 1$. We will say with respect to the vertex x; Q dominates x in G or x is dominated by Q. Moreover, $V(G)$ is a dominating set of G. A *minimal dominating set* Q of a graph G is a dominating set that contains no dominating set of G as a proper subset. Throughout this paper for convenience we will write a md-set of G instead of a minimal dominating set of G. Let us denote by $\mathrm{NMD}(G)$ the total number of md-sets in the graph G. Let x be an arbitrary vertex of $V(G)$. We denote by $\mathcal{Q}_x$ the family of all md-sets Q of G such that $x \in Q$. By $\mathcal{Q}_{-x}$ we denote the family of all md-sets Q of G such that $x \notin Q$. Of course, $\mathcal{Q} = \mathcal{Q}_{-x} \cup \mathcal{Q}_x$ is the family of all md-sets of G and $\mathrm{NMD}(G) = |\mathcal{Q}| = |\mathcal{Q}_{-x}| + |\mathcal{Q}_x|$.

The concept of domination in graphs has existed in literature for a long time, it is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books [4, 5]. In particular, characterization of trees with some extremal domination properties has recently been considered in a number of papers, for instance [1, 3, 6, 7, 8].

In what follows T stands for a tree with the vertex set $V(T)$, $|V(T)|$ denotes the cardinality of $V(T)$. Recall that a vertex of degree 1 is called a *leaf*. For $x \in V(T)$, denote by $L(x)$ the set of leaves attached to the vertex x. If $|L(x)| \geq 1$ then the vertex x we call a *support vertex*. If $|L(x)| = 1$ (resp. $|L(x)| \geq 2$) then the support vertex x we call a *weak support vertex* (resp. a *strong support vertex*). If $L(x) = \{z\}$ then the we say that z is a *single leaf*. Let $V^* \subset V(T)$. Then $L(V^*) = \bigcup_{x \in V^*} L(x)$. If V^* is the set of support vertices of the tree T then $L(V^*)$ is the set of leaves in T. We will use the symbol L to denote the set of leaves in T.

Proposition 1 *If Q is a md-set of T and there exists a support vertex $x \in V(T)$ such that $Q \cap L(x) \neq \emptyset$ then $L(x) \subseteq Q$.*

P R O O F: If $|L(x)| = 1$ then the Proposition is obvious. Assume that $L(x) = \{z_1, \ldots, z_t\}$, $t \geq 2$. Let Q be a md-set of T and assume that there is a support vertex $x \in V(T)$ such that $Q \cap L(x) \neq \emptyset$ and $L(x) \setminus Q \neq \emptyset$. This implies that there is $z_p \in L(x)$ and $z_p \notin Q$. Since $Q \cap L(x) \neq \emptyset$, we see that there is $z_k \in L(x)$, $k \neq p$, and $z_k \in Q$. Consequently, by the minimality of Q, we have that $x \notin Q$. Hence Q does not dominate the vertex z_p in T, which contradicts the fact that Q is dominating set of T. Thus the proof is complete. □

In this paper we consider md-sets Q of T such that for every support vertex $x \in V(T)$, $Q \cap L(x) \neq \emptyset$. Then Proposition 1 implies that $L \subseteq Q$. In the other words we consider md-sets including L. It is clear that every n-vertex tree with $n \geq 3$ has a md-set including the set L. Let us denote by $\mathrm{NMD}_L(T)$ the total number of all md-sets of T including the set of leaves. In this paper we study the number $\mathrm{NMD}_L(T)$. In particular, we characterize trees with the smallest number and the largest number of these sets.

It is easily seen that $\mathrm{NMD}_L(T) = \mathrm{NMD}(T \setminus N[L])$. Let $\mathcal{Q}_L$ be the family of all md-sets of T including the set L. Then $|\mathcal{Q}_L| = |\mathcal{Q}'|$, where $\mathcal{Q}'$ is the family of all md-sets of $T \setminus N[L]$.

Let x be an arbitrary vertex of $V(T)$. We will denote by $\mathcal{Q}_{L,x}$ the family of all md-sets Q of T including the set of leaves L and such that $x \in Q$. Let $\mathcal{Q}_{L,-x}$ denote the family of all md-sets Q of T including the set of leaves L and such that $x \notin Q$. Evidently, $\mathcal{Q}_{L,x} \cup \mathcal{Q}_{L,-x} = \mathcal{Q}_L$ is the family of all md-sets including L. Then the basic rule for counting of md-sets including the set of leaves L in a tree T is as follows $\mathrm{NMD}_L(T) = |\mathcal{Q}_L| = |\mathcal{Q}_{L,x}| + |\mathcal{Q}_{L,-x}|$.

Consider now a collection of n computers which are linked in a network with a structure of a tree. Assume that Q is the subset of the set of computers which control doing some computational problem. The control is possible if there exists direct link of a computer not being in Q to at least one of computer from the set Q. Moreover, it is desired that every computer which is linked to exactly one another computer is the element of the set Q and the set Q is minimal with respect to set inclusion. In this process of the control every computer of the set Q, which is linked to exactly one another computer (not being in Q), piles data from other computers. The question is: what is the smallest number and the largest number of possibilities that we choose computers from the set of n computers to the set Q and what is the structure of the network with such chosen computers. We can reformulate this problem as follows. Let $T = (V, E)$ be a tree with V representing the set of computers, $|V(T)| = n$, and with E representing links between these computers. Therefore every md-set containing all leaves of the tree T represents the set Q of computers which fulfills above mentioned conditions. The total number of all md-sets of T including the set of leaves is equal to the number of all possibilities of the choice such sets Q.

2 Trees with the smallest number of md-sets including L

It is clear that if $|V(T)| = 2$ then $\mathrm{NMD}_L(T) = 0$. We consider n-vertex trees with $n \geq 3$.

Theorem 2 *Let T be an n-vertex tree with $n \geq 3$. Then* $\mathrm{NMD}_L(T) \geq 1$ *with equality if and only if for every two adjacent vertices of T at least one of them is a support vertex.*

P R O O F: Let T be an n-vertex tree with $n \geq 3$ and $L \subset V(T)$ be the set of leaves. The inequality is obvious. Denote $T' = T \setminus N[L]$. Let $x, y \in V(T)$ be arbitrary adjacent vertices of T. Assume that at least one of them, say x, is a support vertex, i.e. $L(x) \neq \emptyset$. We shall prove that $\mathrm{NMD}_L(T) = 1$. Let Q be an arbitrary md-set of T including the set L. Then $L \subseteq Q$. Moreover, by the minimality of the set Q, we have that $N(L) \cap Q = \emptyset$. From the assumption of T and the vertex x we deduce that $x \in N(L)$. This means that T' is totally disconnected. Consequently, $Q = L \cup V(T')$ is the unique md-set of T including the set L. Conversely, assume now that $\mathrm{NMD}_L(T) = 1$ and let Q be the unique md-set of T including L. Of course, $Q = L \cup Q^*$, where Q^* is the unique md-set of T'. On the contrary, suppose that there are adjacent vertices $x, y \in V(T)$ and $L(x) = L(y) = \emptyset$. This implies that $N[L] \cap \{x, y\} = \emptyset$. Clearly, $\{x, y\} \in E(T')$. Let $\mathcal{Q}'$ be a family of all md-sets of T'. Then, by the basic rule for counting md-sets, we have that $|\mathcal{Q}'| = |\mathcal{Q}_x| + |\mathcal{Q}_{-x}|$. Evidently, $|\mathcal{Q}_x| \geq 1$ and $|\mathcal{Q}_{-x}| \geq 1$, so $\mathrm{NMD}(T') = |\mathcal{Q}_x| + |\mathcal{Q}_{-x}| \geq 2$. This contradicts the fact that Q is the unique md-set of T' including L. □

Theorem 3 *Let T be an n-vertex tree with $n \geq 3$. Then* $\mathrm{NMD}_L(T) = 2$ *if and only if the unique maximal connected subtree $T^\star$ of T such that* $L(V(T^\star)) = \emptyset$ *is a star $K_{1,p}$, $p \geq 1$.*

P R O O F: Let T be an n-vertex tree with $n \geq 3$ and $L \subset V(T)$ be the set of leaves of T. Let $K_{1,p}$, $p \geq 1$, be the unique maximal connected subtree of T such that $L(V(K_{1,p})) = \emptyset$. Assume that Q is an arbitrary md-set of T including L. Of course, $L \subseteq Q$. By the general rule for counting the number $\mathrm{NMD}_L(T)$ we have that $\mathrm{NMD}_L(T) = \mathrm{NMD}(T \setminus N[L])$. Denote $T' = T \setminus N[L]$. It is clear that T' is the union of the star $K_{1,p}$, $p \geq 1$, and of totally disconnected graph. Consequently, $Q = L \cup Q'$ where Q' is an arbitrary md-set of T'. Since $\mathrm{NMD}(K_{1,p}) = 2$, we have exactly two sets Q' which gives that $\mathrm{NMD}(T') = 2 = \mathrm{NMD}_L(T)$. Conversely, suppose that $\mathrm{NMD}_L(T) = 2$. Let Q be an arbitrary md-set of T including L. Of course, $L \subseteq Q$, hence $N(L) \cap Q = \emptyset$. By the basic rule for counting sets Q we have that $Q = L \cup Q'$ where Q' is a md-set of the graph T' and $\mathrm{NMD}_L(T) = \mathrm{NMD}(T') = 2$. Let $\widetilde{T}$ be a subgraph of T' such that $T' \setminus \widetilde{T}$ is totally disconnected. Then $\mathrm{NMD}(\widetilde{T}) = \mathrm{NMD}(T') = \mathrm{NMD}_L(T) = 2$. We shall prove that $\widetilde{T} = K_{1,p}$, $p \geq 1$. If $\widetilde{T}$ is connected then, by Theorem **??**, we obtain that $\widetilde{T} = K_{1,p}$, $p \geq 1$. Assume now that $\widetilde{T}$ is disconnected and $\widetilde{T} \neq K_{1,p}$, $p \geq 1$. Let $\widetilde{T}_1$, $\widetilde{T}_2$ be connected components of $\widetilde{T}$. Under the above assumptions about $\widetilde{T}$, let $\{x, y\} \in E(\widetilde{T}_1)$ and $\{u, v\} \in E(\widetilde{T}_2)$. Let

$\mathcal{Q}_1$ and $\mathcal{Q}_2$ be families of all md-sets of $\widetilde{T}_1$ and $\widetilde{T}_2$, respectively. Evidently, $|\mathcal{Q}_1| = |\mathcal{Q}_x| + |\mathcal{Q}_{-x}| \geq 2$ and $|\mathcal{Q}_2| = |\mathcal{Q}_u| + |\mathcal{Q}_{-u}| \geq 2$. Hence, by fundamental combinatorial statements, $\mathrm{NMD}(\widetilde{T}) \geq 4$. Consequently, $\mathrm{NMD}_L(\widetilde{T}) \geq 4$, which contradicts the assumption.

Finally from the above it follows that the unique maximal connected subtree $T^\star$ of T such that $L(V(T^\star)) = \emptyset$ is a star $K_{1,p}$, $p \geq 1$.

Thus the Theorem is proved. □

It is easy to observe

Theorem 4 For an arbitrary $n \geq 3$ there is no an n-vertex tree T with $\mathrm{NMD}_L(T) = 3$. □

3 Trees with the largest number of md-sets including the set of leaves

In this section we determine the largest number of md-sets including the set of leaves in trees. We also characterize trees with the largest number of md-sets including the set L.

A vertex $x \in V(T)$ is *penultimate* if x is not a leaf and x is adjacent to at least $deg_T\, x - 1$ leaves. Note that x is adjacent to $deg_T\, x$ leaves if and only if x is the center of a star $K_{1,n-1}$.

Theorem 5 [9] *Every n-vertex tree T with $n \geq 3$ has a penultimate vertex.*

Theorem 6 *Let x be a strong support vertex of T. Assume that $L(x)$ is the set of leaves attached to the vertex x and $L'(x)$ is an arbitrary proper subset of $L(x)$. Then*

$$\mathrm{NMD}_L(T) = \mathrm{NMD}_L(T \setminus L'(x)).$$

P R O O F: Let $x \in V(T)$ be a strong support vertex and $L(x) = \{z_1, \ldots, z_k\}$, $k \geq 2$. Let $L'(x) \subset L(x)$. Denote $T' = T \setminus L'(x)$. Since $L'(x)$ is a proper subset of $L(x)$, we see that there is a subset $L''(x) = L(x) \setminus L'(x)$ being the set of leaves in T and T', simultaneously. Hence x is the neighbor of every $z_p \in L''(x)$, $p \in \mathcal{I} \subset \{1, \ldots, k\}$. Assume that $\mathcal{Q}_L$ and $\mathcal{Q}'_L$ are families of all md-sets in T and T' including the set L, respectively. It is easily seen that for every $Q \in \mathcal{Q}_L$ holds $x \notin Q$. Hence $|\mathcal{Q}_L| = |\mathcal{Q}_{L,-x}|$. Since $L(x) \subseteq Q$, it follows that $L'(x) \subset Q$. This means that $Q \setminus L'(x) \in \mathcal{Q}'_L$. From the above we have that $|\mathcal{Q}_L| \leq |\mathcal{Q}'_L|$. Conversely, assume that $Q' \in \mathcal{Q}'_L$ is a md-set of T' including the set L. Of course, $L''(x) \subset Q'$ and $x \notin Q'$. Hence $|\mathcal{Q}'_L| = |\mathcal{Q}'_{L,-x}|$. Since

$L'(x)$ is a proper subset of $L(x)$, we conclude that $z_p \in Q'$ for every $p \in \mathcal{I}$, and $Q' \cup L'(x) \in \mathcal{Q}_L$. Consequently, $|\mathcal{Q}'_L| \leq |\mathcal{Q}_L|$. All this together gives that $|\mathcal{Q}_L| = |\mathcal{Q}'_L|$, so $\mathrm{NMD}_L(T) = \mathrm{NMD}_L(T \setminus L'(x))$, which completes the proof. □

Theorem 7 *Let $x \in V(T)$ be a weak support vertex of T with $L(x) = \{z\}$. Assume that x is not a penultimate vertex. Then* $\mathrm{NMD}_L(T) \leq \mathrm{NMD}_L(T \setminus \{z\})$.

P R O O F: Let $x \in V(T)$ be a weak support vertex of T and z be the single leaf attached to the vertex x. Denote $T' = T \setminus \{z\}$. Assume that $\mathcal{Q}_L$ and $\mathcal{Q}'_L$ are families of all md-sets including the set of leaves in T and T', respectively. It is clear that for every $Q \in \mathcal{Q}_L$ we have that $x \notin Q$. Hence $|\mathcal{Q}_L| = |\mathcal{Q}_{L,-x}|$. Since x is not a penultimate vertex of T, it follows that x is not a leaf in T'. By assumptions we have that for every $Q \in \mathcal{Q}_L$, $Q \setminus \{z\} \in \mathcal{Q}'_L$ or $Q \setminus \{z\} \cup \{x\} \in \mathcal{Q}'_L$. Hence $|\mathcal{Q}_L| \leq |\mathcal{Q}'_L|$, which ends the proof. □

Theorem 8 *Let T be an n-vertex tree, $n \geq 5$, $T \neq K_{1,n-1}$. Let $x \in V(T)$ be a strong support vertex with $L(x) = \{z_1, \ldots, z_p\}$, $p \geq 2$. Assume that u is a penultimate vertex of T and $v \in N(u) \setminus L(u)$. Then* $\mathrm{NMD}_L(T) \leq \mathrm{NMD}_L(sub_{\{u,v\}}(T \setminus \{z_i\}))$ *for an arbitrary $1 \leq i \leq p$.*

P R O O F: Let $x \in V(T)$ be a strong support vertex and $L(x) = \{z_1, \ldots, z_p\}$, $p \geq 2$. Assume that z_i, $1 \leq i \leq p$, is a fixed vertex of $L(x)$. Then, by Theorem 6, we have that $\mathrm{NMD}_L(T) = \mathrm{NMD}_L(T \setminus \{z_i\})$ for $1 \leq i \leq p$. Let u be a penultimate vertex of T and $v \in N(u) \setminus L(u)$. The existence of the vertex v gives the fact that $T \neq K_{1,n-1}$. Denote $T' = sub_{\{u,v\}}(T \setminus \{z_i\})$. Let $\mathcal{Q}_L$ and $\mathcal{Q}'_L$ be families of all md-sets including the set of leaves in T and T', respectively. By the basic rule for counting md-sets including L we have that $\mathrm{NMD}_L(T') = |\mathcal{Q}'_{L,u}| + |\mathcal{Q}'_{L,-u}|$. Since u is the penultimate vertex of T, hence by the definition of subdivision of edge, it follows that u is the penultimate vertex in T', too. Let $Q' \in \mathcal{Q}'_L$ be an arbitrary md-set of T' including L. Then it is obvious that $u \notin Q'$. Hence $\mathrm{NMD}_L(T') = |\mathcal{Q}'_{L,-u}|$. Let w be a vertex inserting into the edge $\{u, v\}$. Of course, $|\mathcal{Q}'_{L,-u}| = |\mathcal{Q}'_{L,-w}| + |\mathcal{Q}'_{L,w}|$. Consider two cases.

(1) $w \notin Q'$

Then it is clear that $v \in Q$, so $|\mathcal{Q}'_{L,-w}| = |\mathcal{Q}'_{L,v}| = |\mathcal{Q}_{L,v}|$.

(2) $w \in Q'$

Since $deg_{T'} w = 2$, we have $v \notin Q'$. Therefore, by the definition of subdivision of edge $\{u, v\}$, we obtain that $|\mathcal{Q}'_{L,w}| = |\mathcal{Q}'_{L,-v}| \geq |\mathcal{Q}_{L,-v}|$.

Consequently from the above we have that

$\mathrm{NMD}_L(T') = |\mathcal{Q}'_{L,-u}| = |\mathcal{Q}'_{L,-w}| + |\mathcal{Q}'_{L,w}| \geq |\mathcal{Q}_{L,v}| + |\mathcal{Q}_{L,-v}| = \mathrm{NMD}_L(T)$, which ends the proof. □

Proving analogously as in Theorem 8 we obtain.

Theorem 9 *Let T be an n-vertex tree, $n \geq 5$, and $T \neq K_{1,n-1}$. Assume that $x \in V(T)$ is a weak support vertex of T with $L(x) = \{z\}$ and x is not penultimate. Let u be a penultimate vertex of T and $v \in N(u) \setminus L(u)$. Then* $\mathrm{NMD}_L(T) \leq \mathrm{NMD}_L(sub_{\{u,v\}}(T \setminus \{z\}))$. □

Let $\widetilde{T}$ be an arbitrary tree. From now on for a tree T with $|V(T)| \geq 3$ by $\widetilde{T}$-*addition* we mean a local augmentation which is the operation $T \mapsto \mathrm{ad}_{\widetilde{T}(x,y)}(T)$ of adding to the vertex $x \in V(T)$ a graph $\widetilde{T}$ so that a vertex x is identified with a fixed vertex $y \in V(\widetilde{T})$.

Theorem 10 *Let T be an n-vertex tree with $x \in V(T)$ and let P_t, P_m, $t, m \geq 3$, be subtrees of T attached to the vertex x. Then* $\mathrm{NMD}_L(T) \leq \mathrm{NMD}_L(\mathrm{ad}_{P_t(u,x)}(T \setminus (P_t \setminus \{x\})))$ *where u is the leaf of P_m which is identified with the initial vertex x of P_t.*

P R O O F: Let T be as in the statements of the Theorem. Denote $T' = \mathrm{ad}_{P_t(u,x)}(T \setminus (P_t \setminus \{x\}))$. Let $\mathcal{Q}_L$ and $\mathcal{Q}'_L$ be families of all md-sets including L in T and T', respectively. By the general rule for counting of minimal dominating sets including L we have that $\mathrm{NMD}_L(T) = |\mathcal{Q}_{L,x}| + |\mathcal{Q}_{L,-x}|$. Let $Q \in \mathcal{Q}_L$. Denote $Q^* = Q \cap V(T \setminus (P_t \cup P_m))$, $Q_1 = Q \cap V(P_m)$ and $Q_2 = Q \cap V(P_t)$. Clearly, if $x \in Q$ then $x \in Q_1 \cap Q_2$. Consider the following cases.
(1) $x \in Q$
Then $m, t \geq 3$. Otherwise, since $L \subseteq Q$, we have that Q is not minimal. Identifying vertex x of P_t with the leaf u of P_m we obtain that $Q = Q^* \cup Q_1 \cup Q_2$ is a md-set of T' including the set L. Hence $|\mathcal{Q}_{L,x}| \leq |\mathcal{Q}'_{L,x}|$.

(2) $x \notin Q$
Let $v \in N(x) \cap V(P_t)$ and consider two possibilities.
(2.1) $v \in Q$
In this case the following set $Q^* \cup Q_1 \cup (Q_2 \setminus \{v\})$ or $Q^* \cup Q_1 \cup (Q_2 \setminus \{u\})$ or $Q^* \cup Q_1 \cup Q_2$ is a md-set including the set of leaves in T'.
(2.2) $v \notin Q$
Then $Q = Q^* \cup Q_1 \cup Q_2$ is a md-set including L in T'.

Consequently, from the above and by fundamental combinatorial statements we have that $|\mathcal{Q}_{L,-x}| \leq |\mathcal{Q}'_{L,-x}|$.

Finally, $\mathrm{NMD}_L(T) = |\mathcal{Q}_{L,x}| + |\mathcal{Q}_{L,-x}| \leq |\mathcal{Q}'_{L,x}| + |\mathcal{Q}'_{L,-x}| = \mathrm{NMD}_L(T')$. Thus the Theorem is proved. □

The main results of this section follows.

Theorem 11 *Let T be an n-vertex tree with $n \geq 3$. Then* $\mathrm{NMD}_L(T) \leq \mathrm{NMD}_L(P_n)$.

P R O O F: If $T = K_{1,p}$, $p \geq 2$, then the Theorem is obvious. Assume that $T \neq K_{1,p}$, $p \geq 2$. To avoid trivialities, suppose that $n \geq 5$ and $T \neq P_n$. Let $X \subset V(T)$ be the set of strong support vertices in T. Theorem 8 now shows that there is an n-vertex tree $\widetilde{T}$ such that $\mathrm{NMD}_L(\widetilde{T}) \geq \mathrm{NMD}_L(T)$ and for every $x \in V(\widetilde{T})$, $|L(x)| \leq 1$. Let $Y \subset V(\widetilde{T})$ be the set of weak support vertices y such that y is not penultimate. Then Theorem 9 gives that we can construct an n-vertex tree $T^\star$ with $\mathrm{NMD}_L(T^\star) \geq \mathrm{NMD}_L(\widetilde{T}) \geq \mathrm{NMD}_L(T)$ such that $T^\star$ does not have strong support vertices and every weak support vertex is penultimate. Moreover, using Theorem 10 step by step, we obtain that $\mathrm{NMD}_L(P_n) \geq \mathrm{NMD}_L(T^\star) \geq \mathrm{NMD}_L(T)$, which ends the proof. $\square$

At the end we calculate the number $\mathrm{NMD}_L(P_n)$ being the largest number of md-sets including L among all n-vertex trees. We describe this number using the recurrence relations.

Let P_n be a graph with the vertex set $V(P_n) = \{x_1, \ldots, x_n\}$, $n \geq 1$, numbered in the natural fashion. We denote by $f(n)$, $n \geq 8$, the total number of md-sets of P_n containing vertices x_1, x_{n-4}, x_{n-3} and x_n.

Theorem 12 *Let $n \geq 0$ be an integer. Then for $n \geq 9$*

$$\mathrm{NMD}_L(P_n) = \mathrm{NMD}_L(P_{n-2}) + \mathrm{NMD}_L(P_{n-3}) + f(n)$$
$$\textit{and } f(n) = \mathrm{NMD}_L(P_{n-7}) + f(n-4)$$

with initial conditions

$$\mathrm{NMD}_L(P_0) = \mathrm{NMD}_L(P_2) = 0,$$
$$\mathrm{NMD}_L(P_n) = 1 \textit{ for } n = 1, 3, 4, 5,$$
$$\mathrm{NMD}_L(P_6) = \mathrm{NMD}_L(P_7) = 2, \mathrm{NMD}_L(P_8) = 4 \textit{ and } f(n) = 0 \textit{ for } n =, 5, 6, 7,\ f(8) = 1.$$

P R O O F: The initial conditions for $\mathrm{NMD}_L(P_n)$ are obvious. Let $\mathcal{Q}_L$ be the family of all md-sets of P_n including the set $\{x_1, x_n\}$. Assume that $Q \in \mathcal{Q}_L$. This means that $x_{n-1} \notin Q$. Consider the following cases.

(1) $x_{n-2} \in Q$

Let $\mathcal{Q}_1 \in \mathcal{Q}_L$ be a subfamily of $\mathcal{Q}_L$ such that for $Q \in \mathcal{Q}_1$ we have that $\{x_1, x_{n-2}, x_n\} \subset Q$. Then $x_{n-3} \notin Q$ and $Q = Q^* \cup \{x_n\}$, where Q^* is an arbitrary md-set of graph $P_n \setminus \{x_n, x_{n-1}\}$ including vertices x_1 and x_{n-2}. Since $P_n \setminus \{x_n, x_{n-1}\}$ is isomorphic to P_{n-2}, we have $|\mathcal{Q}_1| = \mathrm{NMD}_L(P_{n-2})$.

(2) $x_{n-2} \notin Q$
Let $\mathcal{Q}_2 \subset \mathcal{Q}_L$ and for arbitrary $Q \in \mathcal{Q}_2$, $x_{n-2} \notin Q$. Then it is easily seen that $x_{n-3} \in Q$ and two possibilities should be considered.

(2.1) $x_{n-4} \in Q$
Let $\mathcal{Q}_2' \subset \mathcal{Q}_2$ and $Q \in \mathcal{Q}_2'$. Under our assumptions $x_1, x_{n-3}, x_{n-4}, x_n \in Q$, so $|\mathcal{Q}_2'| = f(n)$. The initial conditions for $f(n)$ are obvious. Let $n \geq 9$. To calculate the number $f(n)$, consider two possibilities.

(2.1.1) $x_{n-8} \in Q$
In this case Q has the form $Q = Q^* \cup \{x_n, x_{n-3}\}$, where Q^* is an arbitrary md-set of $P_n \setminus \{x_{n-i};\, i = 0, \ldots, 3\}$, isomorphic to P_{n-4}, and contains vertices $x_1, x_{n-4}, x_{n-7}, x_{n-8}$. Therefore, due to the definition of the number $f(n)$, we have $f(n-4)$ sets Q in this case.

(2.1.2) $x_{n-8} \notin Q$
Then $Q = Q^{**} \cup \{x_n, x_{n-3}, x_{n-4}\}$, where Q^{**} is an arbitrary md-set of $P_n \setminus \{x_{n-i};\, i = 0, \ldots, 6\}$, which is isomorphic to P_{n-7} and $x_1, x_{n-7} \in Q^{**}$. Hence we have $\mathrm{NMD}_L(P_{n-7})$ sets in this case.

Consequently, for $n \geq 9$ we have that $f(n) = f(n-4) + \mathrm{NMD}_L(P_{n-7})$.

(2.2) $x_{n-4} \notin Q$
In this case $Q = Q'' \cup \{x_n\}$, where Q'' is an arbitrary md-set of graph $P_n \setminus \{x_n, x_{n-1}, x_{n-2}\}$ and $x_1, x_{n-3} \in Q''$. Since $P_n \setminus \{x_n, x_{n-1}, x_{n-2}\}$ is isomorphic to P_{n-3}, it follows that we have $\mathrm{NMD}_L(P_{n-3})$ sets Q in this case.
Therefore from the above cases we obtain $|\mathcal{Q}_2| = f(n) + \mathrm{NMD}_L(P_{n-3})$ and $f(n) = \mathrm{NMD}_L(P_{n-7}) + f(n-4)$, $n \geq 9$.

Finally for $n \geq 9$, $\mathrm{NMD}_L(P_n) = |\mathcal{Q}_1| + |\mathcal{Q}_2| = \mathrm{NMD}_L(P_{n-2}) + \mathrm{NMD}_L(P_{n-3}) + f(n)$ and $f(n) = \mathrm{NMD}_L(P_{n-7}) + f(n-4)$. This ends the proof. □

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