Int. J. Contemp. Math. Sciences, Vol. 4, 2009, no. 30, 1473 - 1478

On an Equivalence of Simplicial Polytopes

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Abstract

We show that a simplicial isomorphism of boundary polyhedra of simplicial polytopes extends to an equivalence of polytopes. Using this result we present another proof of the well known result that there are exactly five platonic solids.

Mathematics Subject Classification: Primary 57Q15, 52B70; Secondary 57N05

Keywords: Platonic Solids, Combinatorial 2- manifolds, Degree-regular triangulations

1 Introduction and Definitions

This article is an attempt to present a yet another simple proof of the well known classical theorem about classification of Platonic solids. Five geometrical objects, namely, *the Tetrahedra, the Cube, the Octahedra, the Dodecahedra* and *the Icosahedra* are called Platonic solids [2]. Extending continuous functions of spheres to the balls which they bound is a common exercise in a Topology graduate curriculum. But, whether such a result exists in the case of Combinatorial Topological category is not known to the author. We present such a result for the case of equivalence of simplicial polytopes here and use it to establish the result about classification. The existence is addressed by standard technique of constructing combinatorial 2-manifolds. We begin with some definitions.

An *abstract n-simplex* σ is a finite set $\{v_1, v_2, \ldots, v_{n+1}\}$. An abstract n-simplex is also referred to as a n simplex. An *abstract simplicial complex* K is a finite collection of abstract n-simplices for some positive integers n, together with all of their subsets. The greatest integer d such that K has a d simplex is called the *dimension* of K. In this article we will be mainly dealing with simplicial complexes of dimension 2. In such a case the elements of sizes $0, 1$ and 2 are called *vertices, edges* and *faces*, respectively.

Let σ be an abstract n-simplex, *i.e.* $\sigma = \{v_1, v_2, \dots, v_{n+1}\}$. Let $\{x_1, x_2, \dots, x_{n+1}\}$ be a set of geometrically independent (*i.e.* the set $\{x_i - x_1 : 2 \le i \le n+1\}$ is linearly independent) points in some Euclidean space $\mathcal{R}^N$. Then, the convex hull $|\sigma|$ of $\{x_1, x_2, \dots, x_{n+1}\}$ is known as *geometric realization* of σ or geometric simplex corresponding to σ. Henceforth in this article we will omit the term abstract from abstract n-simplex and abstract simplicial complexes.

Let K be a simplicial complex with the set of 0 simplices as $V = \{a_1, a_2, \dots, a_m\}$. We can choose a positive integer N sufficiently large and a set of points $U = \{b_1, b_2, \dots, b_m\}$ in $\mathcal{R}^N$ which is in one - to - one correspondence with V such that the convex hull of $\{b_{i_1}, b_{i_2}, \dots, b_{i_{r+1}}\} \subseteq U$ is a r - simplex in $\mathcal{R}^N$ if $\{a_{i_1}, a_{i_2}, \dots, a_{i_{r+1}}\} \subseteq V$ is a r - simplex in K. Then, the union of all the geometric simplices in $\mathcal{R}^N$ corresponding to simplices in K is called the geometric carrier of K and is denoted by $|K|$.

A *simplicial map* between two simplicial complexes K_1 and K_2 is a map T of the vertex set $V(K_1)$ of K_1 into vertex set $V(K_2)$ of K_2 such that $\sigma \in K_1$ implies $T(\sigma) \in K_2$. The map T is an isomorphism if $T \colon V(K_1) \longrightarrow V(K_2)$ is a bijective map such that $T(\sigma)$ is a simplex if and only if σ is a simplex. One may refer to [5] and [6] for further details about simplicial complexes.

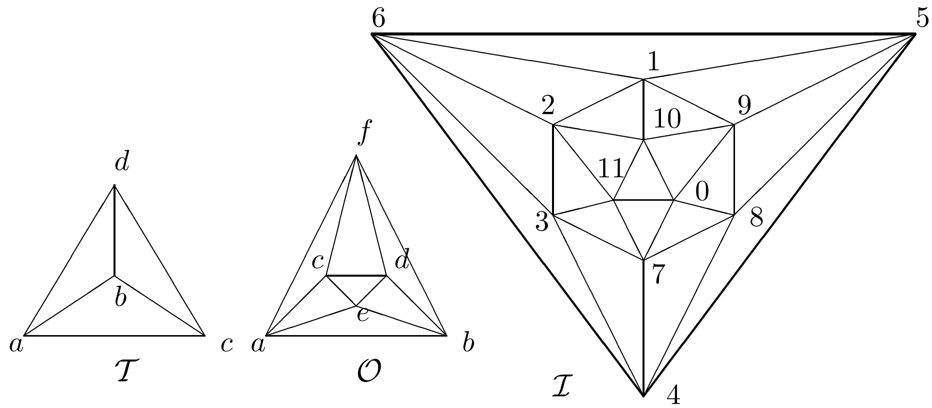
A graph without loops and double edges is an example of 1- dimensional simplicial complex. The number of edges incident with a vertex v in a simplicial complex is called the *degree* of v. A connected finite graph is called a *cycle* if the degree of each vertex is 2. An n-cycle is a cycle on n vertices. It is denoted by $C_n(v_1, v_2, \dots, v_n)$ if the edges are $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ and v_nv_1.

If v is a vertex of a simplicial complex K then the *link* of v in K is the simplicial complex $\mathrm{lk}(v) = \{\sigma \in K : v \notin \sigma, \{v\} \cup \sigma \in K\}$. A finite 2-dimensional simplicial complex K is called a *combinatorial 2-manifold* if $|K|$ is a topological 2-manifold. It is easy to see that a simplicial complex K is a combinatorial 2-manifold if and only if $\mathrm{lk}(v)$ is a cycle for each vertex v of K.

For an $n-$vertex combinatorial 2-manifold M let v, e and f denote the numbers of vertices, edges and faces respectively. Then, the integer $\chi(M) = v - e + f$ is called the *Euler characteristics* of M. If the degree of each vertex in M is same then it is called *degree-regular*. In following section we present some examples of degree-regular combinatorial 2-manifolds.

2 Examples

In this section we give pictorial examples of some orientable equivelar combinatorial 2-manifolds of Euler characteristics 2 :



3 Results

Let P be a convex subset of $\mathcal{R}^3$. A point $x \in P$ is called a *vertex point* (see [4], pp. 17) of P if $y, z \in P, 0 < \lambda < 1$ and $x = \lambda\, y + (1-\lambda)\, z$ imply $x = y = z$. The set of all vertex points of P is called the *vertex set* of P and is denoted as $vertP$. A compact convex set $P \subset \mathcal{R}^3$ is called a *polytope* if $vertP$ is a finite set. Thus, the platonic solids are examples of polytopes.

We say that a hyperplane $H = \{x \in \mathcal{R}^3 : \langle x, u\rangle = a, ||u|| = 1 \text{ and } a \neq 0\}$, *supports* P if H does not intersect P but the distance $\delta(H, P)$ between P and H is 0, where $\delta(H, P) = inf\{||b - a|| : a \in H, b \in P\}$. A set $F \subseteq P$ is called a *face* of P if either $F = \emptyset$ or $F = P$ or there exists a supporting hyperplane H of P such that $F = P \cap H$ (see [4], pp. 17). The faces other than $\emptyset$ and K are called proper faces of P. If all the proper faces of P are simplices then P is called a *simplicial polytope*.

An equivalence of two polytopes P_1 and P_2 (see [4], pp. 38) is a one - to - one map T between the sets $\{F\}$ and $\{F'\}$ of faces of P_1 and P_2 respectively, which preserves inclusions. *i.e.* if $F_1 \subset F_2$ then $T(F_1) \subset T(F_2)$.

In Theorem 1 we show that a simplicial isomorphism of boundary polyhedra of a simplicial polytope extends to an equivalence of polytopes.

Theorem 1. *Let P_1 and P_2 be two simplicial polytopes. Let $T\colon Bd(P_1) \longrightarrow Bd(P_2)$ be an isomorphism of the boundary polyhedra $Bd(P_1)$ and $Bd(P_2)$ of P_1 and P_2 respectively. Then T extends to an equivalence $\tilde{T}\colon P_1 \longrightarrow P_2$.*

Proof. By definition of T, $T(F)$ is a face of P_2 if and only if F is a face of P_1, for all proper faces of P_1 and all the proper faces of P_2 are of this type. Now, extend T by defining $\tilde{T}$ as $\tilde{T}(F) = T(F)$ for all proper faces F of P_1 and

$\tilde{T}(\emptyset) = \emptyset$, $\tilde{T}(P_1) = P_2$. $\tilde{T}$ is a well defined map and is a bijection of set of faces of P_1 onto the set of faces of P_2.

Now, consider two faces F_1 and F_2 of P_1 for which $F_1 \subseteq F_2$. If $F_2 = P_1$ or $F_1 = \emptyset$ then $\tilde{T}(F_1) \subseteq \tilde{T}(F_2)$ is true. Otherwise, let us assume that there exists a $z \in \tilde{T}(F_1) = T(F_1)$ such that $z \notin \tilde{T}(F_2) = T(F_2)$. Then, $T^{-1}(z) \in T^{-1}(T(F_1))$ but $T^{-1}(z) \notin T^{-1}(T(F_2))$. In other words, $T^{-1}(z) \in F_1$ but $T^{-1}(z) \notin F_2$. This is not possible. Hence, $\tilde{T}(F_1) \subseteq \tilde{T}(F_2)$. Thus, $\tilde{T}$ is an equivalence. □

For an $n-$vertex combinatorial 2-manifold M, if the degree of each vertex is d then $nd = 2e = 3f$. So, nd is divisible by 6. Also, $\chi(M) = n - e + f = n - \frac{nd}{2} + \frac{nd}{3} = \frac{n(6-d)}{6}$. Now, if M is connected and $\chi(M)$ is positive then $\chi(M) = 1$ or 2. In that case $(n, d) = (4, 3), (6, 4), (6, 5)$ or $(12, 5)$. For each $(n, d) \in \{(4, 3), (6, 4), (6, 5)\}$ there exists unique (see [3]) combinatorial 2-manifold, namely the 4-vertex 2-sphere, the boundary of the Octahedron and the 6 vertex real projective plane. In Lemma 1 we present a simple proof of the fact that the combinatorial 2-manifold corresponding to the boundary of Icosahedron is the unique degree-regular 12-vertex combinatorial 2-manifold of Euler characteristics 2.

Lemma 1. *Let M be a combinatorial $2-$manifold on 12 vertices. If the degree of each vertex in M is 5, then M is isomorphic to $\mathcal{I}$ given in section 2 above.*

Proof. Let the vertex set V of M be $\{0, 1, \ldots, 11\}$. Assume without loss of generality that $\mathrm{lk}(0) = C_5(1, 2, 3, 4, 5)$. So, $\mathrm{lk}(1)$ has the form $C_5(5, 0, 2, x, y)$, for some $x, y \in V$. Clearly, $x \notin \{0, 1, 2, 3, 5\}$.

If $x = 4$ then $y \notin \{0, 1, 2, 4, 5\}$. If $y \neq 3$, then $\mathrm{lk}(4)$ contains 6 vertices, which is not possible. Hence $y = 3$. Then $\mathrm{lk}(1) = C_5(5, 0, 2, 4, 3)$ and so $\mathrm{lk}(4) = C_5(5, 0, 3, 1, 2)$. This implies $\mathrm{lk}(3) = C_5(2, 0, 4, 1, 5)$, $\mathrm{lk}(2) = C_5(3, 0, 1, 4, 5)$ and $\mathrm{lk}(5) = C_5(1, 0, 4, 2, 3)$. Then M is disconnected. Thus $x \neq 4$. So, we may assume that $x = 6$.

Now, $y \notin \{0, 1, 2, 4, 5, 6\}$. If $y = 3$, then $\mathrm{lk}(3)$ contains 6 vertices, a contradiction. So, we may assume $y = 7$. Then $\mathrm{lk}(1) = C_5(2, 0, 5, 7, 6)$ and $\mathrm{lk}(2)$ has the form $C_5(3, 0, 1, 6, z)$ for some $z \in V$. It is easy to see that $z \in \{5, 7, 8, 9, 10, 11\}$. If $z = 5$ then $\mathrm{lk}(5)$ has > 5 vertices. If $z = 7$ then $C_3(7, 2, 1) \subseteq \mathrm{lk}(6)$, a contradiction. So, we may assume that $z = 8$.

Thus $\mathrm{lk}(2) = C_5(3, 0, 1, 6, 8)$ and $\mathrm{lk}(3)$ has the form $C_5(4, 0, 2, 8, w)$, for some $w \in V$. It is easy to see that $w \in \{6, 7, 9, 10, 11\}$. If $w = 6$ then $C_3(2, 3, 6) \subseteq \mathrm{lk}(8)$, a contradiction. If $w = 7$ then $\mathrm{lk}(7)$ has > 5 vertices. So, we may assume $w = 9$.

Thus $\mathrm{lk}(3) = C_5(4, 0, 2, 8, 9)$ and $\mathrm{lk}(4)$ has the form $C_5(5, 0, 3, 9, u)$. A similar argument as in the previous case, shows that $u = 10$ or 11, say $u = 10$. So, $\mathrm{lk}(4) = C_5(5, 0, 3, 9, 10)$ and hence $\mathrm{lk}(5) = C_5(7, 1, 0, 4, 10)$. Now, $\mathrm{lk}(6)$ has the form $C_5(7, 1, 2, 8, v)$ for some $v \in V$. Since the vertices $0, 1, 2, 3, 4, 5 \notin \mathrm{lk}(11)$, it follows that $6 \in \text{lk}(11)$, *i.e.*, $v = 11$. Hence $\text{lk}(6) = C_5(7, 1, 2, 8, 11)$. This implies $\text{lk}(7) = C_5(11, 6, 1, 5, 10)$, $\text{lk}(10) = C_5(9, 4, 5, 7, 11)$, $\text{lk}(8) = C_5(6, 2, 3, 9, 11)$, $\text{lk}(9) = C_5(11, 8, 3, 4, 10)$ and $\text{lk}(11) = C_5(7, 6, 8, 9, 10)$.

This is isomorphic to $\mathcal{I}$ by the map $T\colon M \longrightarrow \mathcal{I}$ given by $T(0) = 0$, $T(1) = 7$, $T(2) = 8$, $T(3) = 9$, $T(4) = 10$, $T(5) = 11$, $T(6) = 4$, $T(7) = 3$, $T(8) = 5$, $T(9) = 1$, $T(10) = 2$ and $T(11) = 6$.

□

Of all the five polyhedra (see [1] for definitions and details of polyhedra) corresponding to boundary of platonic solids, three are simplicial, namely, the Tetrahedron, the Octahedron and the Icosahedron. It is well known that the Cube and Dodecahedron are dual polyhedron of Octahedron and Icosahedron respectively. Since, the Tetrahedron, the Octahedron and the Icosahedron are the only orientable simplicial degree-regular polyhedron (*i.e.* orientable combinatorial 2-manifolds) of Euler characteristic 2, it follows that up to isomorphism :

Theorem 2. *There are exactly five orientable degree-regular polyhedra of Euler characteristic 2. These are, namely, the Tetrahedron, the Octahedron, the Cube, the Dodecahedron and the Icosahedron.*

It follows from Theorem 1 and Theorem 2 that there are exactly five platonic solids.

4 Acknowledgement

Part of this work was done in the USBAS, GGSIP University, Delhi and IIIT, Allahabad. The author wishes to thank R. P. Shukla who read an earlier version of the article and gave important suggestions for improvements.

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Received: March, 2009