

Uniform Convergence of Schwarz Method for Noncoercive Variational Inequalities

M. Haiour and E. Hadidi

Department of mathematics, Faculty of the Sciences,
University Badji Mokhtar, P.O 23000 Annaba, Algeria.

Abstract

In this paper we study noncoercive variational inequalities studied by Courte-Dumont, using the Schwarz method. The main idea of this method consists in decomposing the domain in two subdomains. We demonstrate that the discretisation on every subdomain converges in uniform norm and we give a result of approximation for the method in uniform norm.

Mathematics Subject Classification: 05C38, 15A15; 05A15, 15A18

Keywords: Schwarz method, Variational inequalities, L^∞ -error estimates.

1 Introduction

We are interested in the following noncoercive variational inequality.

Find $u \in H_0^1(\Omega)$ solution of

$$\begin{cases} a(u, v - u) \geq (f, v - u) \\ u \leq \Psi, v \leq \Psi \end{cases} \quad (1)$$

where Ω is a smooth bounded domain of \mathbb{R}^2 with boundary $\partial\Omega$.

and the noncoercive bilinear form $a(u, v)$.

or equivalently. Find $u \in H_0^1(\Omega)$ solution of

$$\begin{cases} b(u, v - u) \geq (f + \lambda u, v - u) \\ u \leq \Psi, v \leq \Psi \end{cases} \quad (2)$$

where

$$b(u, v) = a(u, v) + \lambda(u, v) \quad (3)$$

and $\lambda > 0$ large enough such that $\forall v \in H_0^1(\Omega)$ we have

$$b(v, v) \geq \mu \|v\|_{H^1(\Omega)}^2, \mu > 0 \quad (4)$$

In section 2, we give the continuous V.I problem, we study the existence and the uniqueness of the solution, then we introduce the continuous Schwarz method. In section 3, we consider the discrete problem and we establish a survey similar to the one of the continuous case. In section 4, we establish the error estimates in the L^∞ norm for the problem studied.

2 The Continuous Problem

2.1 Notations and Assumptions

Let's consider the functions

$$a_{i,j}(x), a_i(x), a_0(x) \in C^2(\overline{\Omega}), x \in \overline{\Omega}, 1 \leq i, j \leq n \quad (5)$$

such that

$$\sum_{1 \leq i, j \leq n} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2; \xi \in \mathbb{R}^n, \alpha > 0 \quad (6)$$

$$a_{ij}(x) = a_{ji}(x); a_0(x) \geq \beta > 0 \quad (7)$$

We define the bilinear form, $\forall u, v \in H_0^1(\Omega)$

$$a(u, v) = \int_{\Omega} \left(\sum_{1 \leq i, j \leq n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1 \leq i \leq n} a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x) u v \right) dx \quad (8)$$

Let f be

$$f \in L^\infty(\Omega) \cap C^2(\overline{\Omega}); f \geq 0 \quad (9)$$

and on Ω

$$K_{(\Psi, g)} = \{v \in H^1(\Omega), v - g \in H_0^1(\Omega), 0 \leq v \leq \Psi\} \quad (10)$$

with the obstacle Ψ and g is a regular function defined on $\partial\Omega$.

$$\Psi, g \in W^{2,p}(\Omega), p > 2; 0 \leq g \leq \Psi \quad (11)$$

2.2 The Continuous Problem

Find $u \in K_{(\Psi,g)}$ the solution of

$$b(u, v - u) \geq (f + \lambda u, v - u), \forall v \in K_{(\Psi,g)} \quad (12)$$

Theorem 2.1 ([10]) Under the conditions (5) to (11), the problem (12) has an unique solution $u \in K_{(\Psi,g)}$. Moreover we have

$$u \in W^{2,p}(\Omega), 2 \leq p < \infty \quad (13)$$

Denote the solution u of the variational inequality (12) by $\sigma(f, g)$.

We introduce the property of the monotonocity of the solution of (12).

Proposition 2.2 Under the hypotheses (5) to (11) and previous notations, we have if $f \leq \tilde{f}$ and $g \leq \tilde{g}$ then $\sigma(f, g) \leq \sigma(\tilde{f}, \tilde{g})$

Proof. Let $f \leq \tilde{f}$ and $g \leq \tilde{g}$.

Let's put $u = \sigma(f, g)$ and $\tilde{u} = \sigma(\tilde{f}, \tilde{g})$, we have

$$a(u, v) \leq (f, v), \forall u \in K_{(\Psi,g)}, v \geq 0$$

therefore

$$a(u, v) \leq (f, v) \leq (\tilde{f}, v), \forall u \in K_{(\Psi,g)}, v \geq 0$$

thus

$$a(u, v) \leq (\tilde{f}, v), \forall u \in K_{(\Psi,g)}, v \geq 0$$

then, u is one subsolution for the solution $w = \sigma(\tilde{f}, \tilde{g})$.

As w is the biggest subsolution, we have

$$u = \sigma(f, g) \leq w = \sigma(\tilde{f}, \tilde{g})$$

One knows that $\sigma(\tilde{f}, g) \leq \sigma(\tilde{f}, \tilde{g})$.

then

$$u = \sigma(f, g) \leq \tilde{u} = \sigma(\tilde{f}, \tilde{g})$$

hence

$$\sigma(f, g) \leq \sigma(\tilde{f}, \tilde{g}).$$

Remark 2.3 If $g = \tilde{g}$ we have $\sigma(f) \leq \sigma(\tilde{f})$.

We show an important proposition. Which give the Continuous dependence to the data g and f contrary to those in [1] that are g and Ψ .

Proposition 2.4 Under the hypotheses (5) to (11) and previous notations, we have

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \leq C \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right) \quad (14)$$

where C is an independent constant of f and g , $Ca_0(x) \geq 1$.

Proof. Let's put

$$\Phi = C \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right)$$

We have

$$\begin{aligned} \tilde{f} &\leq f + \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right) \\ &\leq f + Ca_0 \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right) \\ &\leq f + a_0 \Phi \end{aligned}$$

and

$$\tilde{g} \leq g + C \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right) = g + \Phi$$

As

$$\sigma(f + a_0 \Phi, g + \Phi) = \sigma(f, g) + \Phi$$

we have

$$\sigma(\tilde{f}, \tilde{g}) \leq \sigma(f, g) + \Phi$$

then

$$\sigma(\tilde{f}, \tilde{g}) - \sigma(f, g) \leq \Phi$$

Since (\tilde{f}, \tilde{g}) and (f, g) are symmetrical, we have also

$$\sigma(f, g) - \sigma(\tilde{f}, \tilde{g}) \leq \Phi$$

then

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \leq C \left(\|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)} \right).$$

2.3 The Continuous Schwarz Algorithm

We consider the following problem. Find $u \in H_0^1(\Omega)$ the solution of

$$\begin{cases} b(u, v - u) \geq (f + \lambda u, v - u) \\ u \leq \Psi, v \leq \Psi \end{cases} \quad (15)$$

We decompose Ω into two overlapping polygonal subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \quad (16)$$

In [10], the solution u satisfies the condition of the following local regularity

$$u / \Omega_i \in W^{2,p}(\Omega_i), 2 \leq p < \infty \quad (17)$$

We denote $\partial\Omega_i$ the boundary of Ω_i and

$$\Lambda_1 = \partial\Omega_1 \cap \Omega_2, \Lambda_2 = \partial\Omega_2 \cap \Omega_1 \quad (18)$$

We assume that

$$\overline{\Lambda}_1 \cap \overline{\Lambda}_2 = \emptyset \quad (19)$$

2.3.1 A Continuous Algorithm

Starting from

$$u_1^0 = 0, u_2^0 = \bar{u} \quad (20)$$

such that \bar{u} is a solution of the following equation

$$b(\bar{u}, v) = (f + \lambda \bar{u}, v), \forall v \in K_{(\Psi, 0)} \quad (21)$$

We define the continuous sequence of Schwarz $(u^n)_{n \in \mathbb{N}}$ such that

On Ω_1 , we have

$$\begin{cases} u_1^{n+1} \in K_{(\Psi, 0)} \text{ is a solution of} \\ b_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f_1 + \lambda u_1^n, v - u_1^{n+1}), \forall v \in K_{(\Psi, 0)} \\ u_1^{n+1} = u_2^n \text{ on } \Lambda_1, v = u_2^n \text{ on } \Lambda_1 \end{cases} \quad (22)$$

On Ω_2 , we have

$$\begin{cases} u_2^{n+1} \in K_{(\Psi, 0)} \text{ is a solution of} \\ b_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f_2 + \lambda u_2^n, v - u_2^{n+1}), \forall v \in K_{(\Psi, 0)} \\ u_2^{n+1} = u_1^{n+1} \text{ on } \Lambda_2, v = u_1^{n+1} \text{ on } \Lambda_2 \end{cases} \quad (23)$$

where $f_i = (f + \lambda u^n) / \Omega_i, i = 1, 2$.

and $u_i = u / \Omega_i, i = 1, 2$.

Theorem 2.5 ([12]) *The sequences (u_1^{n+1}) and (u_2^{n+1}) ; $n \geq 0$ of the Schwarz method converge geometrically to the unique solution u of the obstacle problem (15), $\exists \rho \in]0, 1[, \forall n \geq 0$*

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq (\rho)^n \|u^0 - u\|_{L^\infty(\Lambda_i)}, i = 1, 2 \quad (24)$$

3 The Discrete Problem

3.1 Discretization

Let $V_{h_i} = V_{h_i}(\Omega_i)$ be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$.

For $w \in C(\overline{\Lambda}_i)$, we define the following space

$$V_{h_i}^{(w)} = \{v \in V_{h_i} / v = 0 \text{ on } \partial\Omega \cap \Omega_i; v = \pi_{h_i}(w) \text{ on } \Lambda_i\} \quad (25)$$

where π_{h_i} denotes the interpolation operator on Λ_i . For $i = 1, 2$, let τ^{h_i} be a standard regular finite element triangulation in Ω_i , h_i being the meshsize. We suppose that the two triangulations are mutually independent on $\Omega_1 \cup \Omega_2$. A triangle belonging to one triangulation does not necessarily belong to the other. We assume that the corresponding matrices resulting from the discretizations of problems (22) and (23), are M-matrices ([15]).

3.2 Position of The Discrete Problem

The discrete problem is find $u_h \in H_0^1(\Omega)$ the solution of

$$\begin{cases} b(u_h, v_h - u_h) \geq (f + \lambda u_h, v_h - u_h) \\ u_h \leq r_h \Psi, v_h \leq r_h \Psi \end{cases} \quad (26)$$

Theorem 3.1 ([8]) Under the conditions in (5) to (11) and the maximum principle, there exists a constant C independent of h such that

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 |\ln h|^2 \quad (27)$$

3.3 The Discrete Schwarz Method

We give the discrete Schwarz method defined in (22) and (23) as follows.

3.3.1 Discrete Algorithm

Starting from

$$u_{1h}^0 = 0, u_{2h}^0 = \bar{u}_h \quad (28)$$

such that \bar{u}_h is a solution of the following equation

$$b(\bar{u}_h, v) = (f + \lambda \bar{u}_h, v), \forall v \in K_{(\Psi, 0)} \quad (29)$$

We define the discrete sequence of Schwarz $(u_h^n)_{n \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} u_{1h}^{n+1} \in V_{h_1}^{(u_{2h}^n)} \text{ is a solution of} \\ b_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1 + \lambda u_{1h}^n, v - u_{1h}^{n+1}), \forall v \in V_{h_1}^{(u_{2h}^n)} \\ u_{1h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi \end{array} \right. \quad (30)$$

and

$$\left\{ \begin{array}{l} u_{2h}^{n+1} \in V_{h_2}^{(u_{1h}^{n+1})} \text{ is a solution of} \\ b_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2 + \lambda u_{2h}^n, v - u_{2h}^{n+1}), \forall v \in V_{h_2}^{(u_{1h}^{n+1})} \\ u_{2h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi \end{array} \right. \quad (31)$$

Remark 3.2 Zhou in [16] gives the algebraic form of the discrete algorithm and the geometrical convergence of the sequence.

$$\|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq (\theta)^n \|u_h - u_h^0\|_{L^\infty(\Lambda_i)}; 0 < \theta < 1; i = 1, 2. \quad (32)$$

4 L^∞ -Error Estimate

4.1 Auxiliary Sequences

We introduce two discrete auxiliary sequences.

Starting from

$$w_{1h}^0 = 0, w_{2h}^0 = \bar{u}_h \quad (33)$$

such that \bar{u}_h is a solution of the following equation

$$b(\bar{u}_h, v) = (f + \lambda \bar{u}_h, v), \forall v \in K_{(\Psi, 0)} \quad (34)$$

We define the discrete auxiliary sequences $(w_h^n)_{n \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} w_{1h}^{n+1} \in V_{h_1}^{(u_2^n)} \text{ is the solution of} \\ b_1(w_{1h}^{n+1}, v - w_{1h}^{n+1}) \geq (f_1 + \lambda u_{1h}^n, v - w_{1h}^{n+1}), \forall v \in V_{h_1}^{(u_2^n)} \\ w_{1h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi \end{array} \right. \quad (35)$$

and

$$\left\{ \begin{array}{l} w_{2h}^{n+1} \in V_{h_2}^{(u_1^{n+1})} \text{ is the solution of} \\ b_2(w_{2h}^{n+1}, v - w_{2h}^{n+1}) \geq (f_2 + \lambda u_{2h}^n, v - w_{2h}^{n+1}), \forall v \in V_{h_2}^{(u_1^{n+1})} \\ w_{2h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi \end{array} \right. \quad (36)$$

Denote by w_{ih}^{n+1} the finite element approximation of u_i^{n+1} defined in (22) and (23).

We will give the lemma which plays a key role in proving our principal result in this paper.

We specify the following notations.

$$|.|_1 = \|\cdot\|_{L^\infty(\Lambda_1)}, |.|_2 = \|\cdot\|_{L^\infty(\Lambda_2)}$$

$$\|\cdot\|_1 = \|\cdot\|_{L^\infty(\Omega_1)}, \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)}$$

$$\pi_{h_1} = \pi_{h_2} = \pi_h$$

The following lemma given in [1].

Lemma 4.1 *We have*

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)} \leq \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_{L^\infty(\Omega_1)} + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_{L^\infty(\Omega_2)} \quad (37)$$

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_{L^\infty(\Omega_2)} \leq \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_{L^\infty(\Omega_2)} + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_{L^\infty(\Omega_1)} \quad (38)$$

Proof. The demonstration is an adaptation of the one in [1] given for the problem of variational inequality.

But our contribution it is that one demonstrates that this lemma remained true for the problem introduces in this paper, while using a proposition with g and $f + \lambda u$ and either g and Ψ as in [1].

To simplify the notation, one takes

$$h_1 = h_2 = h$$

For $n = 0$, using the discrete form of proposition 2.4, we get

$$\begin{aligned} \|u_1^1 - u_{1h}^1\|_1 &\leq \|u_1^1 - w_{1h}^1\|_1 + \|w_{1h}^1 - u_{1h}^1\|_1 \leq \|u_1^1 - w_{1h}^1\|_1 + |\pi_h u_2^0 - \pi_h u_{2h}^0|_1 \\ &\leq \|u_1^1 - w_{1h}^1\|_1 + |u_2^0 - u_{2h}^0|_1 \leq \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2 \\ \|u_2^1 - u_{2h}^1\|_2 &\leq \|u_2^1 - w_{2h}^1\|_2 + \|w_{2h}^1 - u_{2h}^1\|_2 \leq \|u_2^1 - w_{2h}^1\|_2 + |\pi_h u_1^1 - \pi_h u_{1h}^1|_2 \\ &\leq \|u_2^1 - w_{2h}^1\|_2 + |u_1^1 - u_{1h}^1|_2 \leq \|u_2^1 - w_{2h}^1\|_2 + \|u_1^1 - u_{1h}^1\|_1 \\ &\leq \|u_2^1 - w_{2h}^1\|_2 + \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2 \end{aligned}$$

Therefore

$$\|u_1^1 - u_{1h}^1\|_1 \leq \sum_{p=1}^1 \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^0 \|u_2^p - w_{2h}^p\|_{\Omega_2}$$

$$\|u_2^1 - u_{2h}^1\|_2 \leq \sum_{p=0}^1 \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^1 \|u_1^p - w_{1h}^p\|_1$$

For $n = 1$, using the discrete form of proposition 2.4, we get

$$\begin{aligned}
\|u_1^2 - u_{1h}^2\|_1 &\leq \|u_1^2 - w_{1h}^2\|_1 + \|w_{1h}^2 - u_{1h}^2\|_1 \leq \|u_1^2 - w_{1h}^2\|_1 + |\pi_h u_2^1 - \pi_h u_{2h}^1|_1 \\
&\leq \|u_1^2 - w_{1h}^2\|_1 + |u_2^1 - u_{2h}^1|_1 \leq \|u_1^2 - w_{1h}^2\|_1 + \|u_2^1 - u_{2h}^1\|_2 \\
&\leq \|u_1^2 - w_{1h}^2\|_1 + \|u_2^1 - w_{2h}^1\|_2 + \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2 \\
\|u_2^2 - u_{2h}^2\|_2 &\leq \|u_2^2 - w_{2h}^2\|_2 + \|w_{2h}^2 - u_{2h}^2\|_2 \leq \|u_2^2 - w_{2h}^2\|_2 + |\pi_h u_1^2 - \pi_h u_{1h}^2|_2 \\
&\leq \|u_2^2 - w_{2h}^2\|_2 + |u_1^2 - u_{1h}^2|_2 \leq \|u_2^2 - w_{2h}^2\|_2 + \|u_1^2 - u_{1h}^2\|_1 \\
&\leq \|u_2^2 - w_{2h}^2\|_2 + \|u_1^2 - w_{1h}^2\|_1 + \|u_2^1 - w_{2h}^1\|_2 + \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u_1^2 - u_{1h}^2\|_1 &\leq \sum_{p=1}^2 \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^1 \|u_2^p - w_{2h}^p\|_2 \\
\|u_2^2 - u_{2h}^2\|_2 &\leq \sum_{p=0}^2 \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^2 \|u_1^p - w_{1h}^p\|_1
\end{aligned}$$

Let's suppose that

$$\|u_2^n - u_{2h}^n\|_2 \leq \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^n \|u_1^p - w_{1h}^p\|_1$$

Then, using the discrete form of proposition 2.4, we get

$$\begin{aligned}
\|u_1^{n+1} - u_{1h}^{n+1}\|_1 &\leq \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|w_{1h}^{n+1} - u_{1h}^{n+1}\|_1 \leq \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + |\pi_h u_2^n - \pi_h u_{2h}^n|_1 \\
&\leq \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + |u_2^n - u_{2h}^n|_1 \leq \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|u_2^n - u_{2h}^n\|_2 \\
&\leq \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^n \|u_1^p - w_{1h}^p\|_1
\end{aligned}$$

Therefore, we have

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1$$

Using the previous estimation, we have

$$\begin{aligned}
\|u_2^{n+1} - u_{2h}^{n+1}\|_2 &\leq \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|w_{2h}^{n+1} - u_{2h}^{n+1}\|_2 \leq \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + |\pi_h u_1^{n+1} - \pi_h u_{1h}^{n+1}|_2 \\
&\leq \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + |u_1^{n+1} - u_{1h}^{n+1}|_2 \leq \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1
\end{aligned}$$

$$\leq \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1$$

Finally, we have

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \leq \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1$$

4.2 L^∞ -Error Estimate

We finish by L^∞ - error estimate.

Theorem 4.2 *There exists a constant C independent of both h and n such that*

$$\|u_i - u_{ih}\|_{L^\infty(\Omega_i)} \leq Ch^2 |\ln h|^3; i = 1, 2. \quad (39)$$

Proof. For $i = 1$, we have

$$\|u_1 - u_{1h}\|_1 \leq \|u_1 - u_1^{n+1}\|_1 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1 + \|u_{1h}^{n+1} - u_{1h}\|_1$$

We used theorem 2.5 and remark 3.2

$$\leq (\rho)^{2n} |u^0 - u|_1 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1 + (\theta)^{2n} |u_h - u_h^0|_1$$

and lemma 4.1

$$\leq (\rho)^{2n} |u^0 - u|_1 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + (\theta)^{2n} |u_h - u_h^0|_1$$

and theorem 3.1

$$\leq (\rho)^{2n} |u^0 - u|_1 + 2(n+1) Ch^2 |\ln h|^2 + (\theta)^{2n} |u_h - u_h^0|_1$$

Let's put $(\alpha)^{2n} \leq h^2$

where $\alpha = \max(\rho, \theta)$, therefore we find

$$\|u_1 - u_{1h}\|_{L^\infty(\Omega_1)} \leq Ch^2 |\ln h|^3$$

we get similar result for $i = 2$.

References

- [1] M. Boulbrachene and S. Saadi, Maximum norm analysis of an overlapping nonmatching grids method for the obstacle problem, Hindawi publishing corporation, (2006), 1 - 10.
- [2] M. Boulbrachene, M. Haiour and S. Saadi, L^∞ -estimates for a system of quasivariational inequalities, IJMMS, (2003), 1 - 10.
- [3] M. Boulbrachene, M. Haiour and B. Chentouf, On a noncoercive system of quasi-variational inequalities related to stochastic control problems, Journal of inequalities in pure and applied mathematics, **3** (2002).
- [4] M. Boulbrachene and M. Haiour, The finite element approximation of Hamilton- Jacobi-Bellman equation, Computers and mathematics with applications, **41** (2001), 993 - 1007.
- [5] L. Badea, On the Schwarz alternating method with more than two sub-domains for nonlinear monotone problems, SIAM Journal on Numerical Analysis, **28** (1991), 197 - 204.
- [6] M. Boulbrachene, Ph. Cortey-Dumont and J.-C.Miellou, Mixing finite elements and finite differences in a subdomain method, First International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia,(1988), 198 - 216.
- [7] X.-C. Cai, T. P. Mathew, and M. V. Sarkis, Maximum norm analysis of overlapping nonmatching grid discretizations of elliptic equations, SIAM Journal on Numerical Analysis, **37** (2000), 1709 - 1728.
- [8] Ph. Cortey-Dumont, On finite element approximation in the L^∞ -norm of variational inequalities, Numerische Mathematik, **47**(1985), 45 - 57.
- [9] P. G. Ciarlet and P.-A. Raviart, Maximum principle and uniform convergence for the finite element method, Computer Methods in Applied Mechanics and Engineering, **2** (1973), 17 - 31.
- [10] J.Hannouzet and P. Joly, Convergence uniforme des iteres definissant la solution d'une inéquation quasi-variationnelle, C.R.Acad, Sci, Paris , Serie A, **286** (1978)
- [11] P.-L. Lions, On the Schwarz alternating method. I, First International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia, (1988), 1 - 42.

- [12] P.-L. Lions, On the Schwarz alternating method. II. Stochastic interpretation and order properties, Domain Decomposition Methods, SIAM, Philadelphia, (1989), 47 - 70.
- [13] T. P. Mathew and G. Russo, Maximum norm stability of difference schemes for parabolic equations on overset nonmatching space-time grids, *Mathematics of Computation*, **72** (2003), 619 - 656.
- [14] G. H. Meyer, Free boundary problems with nonlinear source terms, *Numer. Math.*, **43** (1984), 463 - 482.
- [15] J. Zeng and S. Zhou, Schwarz algorithm of the solution of variational inequalities with nonlinear source terms, *Applied Mathematics and Computations*, **97** (1988), 23 - 35.
- [16] J. Zeng and S. Zhou, On monotone and geometric convergence of Schwarz methods for two-sided obstacle problems, *SIAM Journal on Numerical Analysis*, **35** (1998), 600 - 616.

Received: March, 2009