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# On a Subclass of Univalent Uniformly Convex Functions Defined by Certain Linear Operator

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**Abstract**

In this paper a new subclass of uniformly convex functions with negative coefficients defined by Dziok- Srivastava linear operator is introduced. Characterization properties exhibited by certain fractional derivative operators of functions and the result of modified Hadmard product are discussed for this class. Further class preserving integral operator , extreme point and other interesting properties for this class are also indicated .

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## 1. Introduction and Definitions

Let S denote the class of functions of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_n$$

Which are analytic and univalent in the unit disk $U=\{z:|z|<1\}$. Also denote by T the class of the form

(1.2) $$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (z\in U) \quad (a_n\geq 0),$$

which are analytic and univalent in U.
For functions

(1.3) $$f_j\ (z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n \qquad (a_{n,j}\geq 0),\ (j=1,2)$$

in the class T , the modified Hadamard product $f_1 * f_2$ of $f_1(z)$ and $f_2(z)$ is defined by

(1.4) $$(f_1 * f_2)(z)=z-\sum_{n=2}^{\infty}a_{n,1}\ a_{n,2}z^n$$

A function $f(z)\in S$ is said to be $\beta$-uniformly starlike functions of order $\alpha$ denoted by $\beta-S(\alpha)$ iff

(1.5) $$\operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}-\alpha\right\}\geq\beta\left|\frac{z\,f'(z)}{f(z)}-1\right|,$$

for some $\alpha(-1\leq\alpha<1),\beta\geq 0$ and all ,$(z\in U)$ , and is said to be $\beta$-uniformly convex of order $\alpha$ denoted by
$\beta-K(\alpha)$ iff

(1.6) $$\operatorname{Re}\left\{1+\frac{z\,f''(z)}{f'(z)}-\alpha\right\}\ \geq\ \beta\ \left|\frac{z\,f''(z)}{f'(z)}\right|,$$

for some $\alpha(-1\leq\alpha<1),\beta\geq 0$ and all $(z\in U)$ .
The class $0-S(\alpha)=S(\alpha)$, and $0-K(\alpha)=K(\alpha)$, where $S(\alpha)$, and $K(\alpha)$ are respectively the well-known classes of starlike and convex functions of order $\alpha$ $(0\leq\alpha<1)$.
The classes $S(\alpha)$, and $K(\alpha)$ were first studied by Reborston [14] , Schild [1] , Silverman [7] , and others . While the class $\beta-S(\alpha)$ and $\beta-K(\alpha)$ were introduced and studied by Goodman [2] , Ronneing [5] , and Minda and Ma [4].
Let

(1.7)
$$S^*(\alpha)=S(\alpha)\cap T,\quad K^*(\alpha)=K(\alpha)\cap T\ ,\beta-S^*(\alpha)=[\beta-S(\alpha)]\cap T\ ,$$

and
$$\beta-K^*(\alpha)=[\beta-K(\alpha)]\cap T$$

For $\alpha_i\in C(i=1,2,3...,l)$ and $\beta_i\in C-\{0,-1,-2,...\}(j=1,2\ ,3,...,m)$ , the generalized hypergeometric function is defined by

(1.8) $${}_lF_m(\alpha_1\ ...,\alpha_L;\beta_1,...,\beta_m)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n}\cdot\frac{z^n}{n!} \qquad ,(l\leq m+1;m\in N_\circ=\{0,12,...\})$$

where $(\alpha)_n$ is the Pochhammer symbol defined by

(1.9) $$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n=0 \\ a(a+1)(a+2)...(a+n+1) & ,n \in N = 1,2,... \end{cases}$$

Corresponding to the function $h(\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m; z) = z \;{}_lF_m(\alpha_1, ..., \alpha_l \,; \beta_1, ..., \beta_m)$ the Dziok-Srivastava operator [9] , $H_m^l \; (\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m)$ is defined by

(1.10) $$H_m^l \; (\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m) \; f(z) = h(\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m; z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} ... (\alpha_L)_{n-1}}{(\beta_1)_{n-1} ... (\beta_m)_{n-1}} a_n \frac{z^n}{(n-1)!}$$

It is well know [12] that

(1.11) $$\alpha_1 H_m^l \; (\alpha_1 + 1, ..., \alpha_l \,; \beta_1, ..., \beta_m) \quad f(z) = z[H_m^l(\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m) f(z)]'$$

$$+ (\alpha_1 - 1) \; H_m^l(\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m) f(z)$$

To make the notation simple , we write ,

$H_m^l[\alpha_1] \;\; f(z) = H_m^l(\alpha_1 \; ..., \alpha_l \,; \beta_1, ..., \beta_m) f(z)$ .

We note that special cases of the Dziok-Srivastava operator $H_m^l[\alpha_1]$ include the Hohlov linear operator [16] ,the Carlson-Shafer operator [3] , the Ruschweyh derivative operator [21] ,the Srivastava- Owa fractional operators [22] , and many others .

Now using $H_m^l[\alpha_1]$ we define the following subclass of analytic function .

**Definition 1 :**For A , B arbitrary fixed real number , $-1 \le B < \alpha < 1$, a function $f(z) \in T$ defined by (1.2) is said to be in the class $S_m^l[A, B, \alpha, \gamma]$ if it satisfies

(1.12) $$\alpha_1 \frac{H_m^l[\alpha_1 + 1]\phi_\gamma(z)}{H_m^l[\alpha_1]\phi_\gamma(z)} + 1 - \alpha_1 \prec \frac{1 + [(A-B)(1-\alpha) + B]z}{1 + Bz} \quad , (z \in E)$$

Where $, 0 \le \alpha < 1, 0 \le \gamma < 1, n \in N_\circ = N \cup 0$.

The condition (1.12) is equivalent to

(1.13)
$$\left|\frac{\frac{z(H_m^l[\alpha_1]\phi_\gamma(z))'}{H_m^l[\alpha_1]\phi_\gamma(z)}-1}{B+(A-B)(1-\alpha)-B\frac{z(H_m^l[\alpha_1]\phi_\gamma(z))'}{H_m^l[\alpha_1]\phi_\gamma(z)}}\right|<1 \qquad ,z\in E$$

Where

(1.14)
$$\phi_\gamma(z)=(f*S_\gamma)(z)=z-\sum_{n=2}^{\infty}a_nC(\gamma,n)z^n$$

we observe that , by specializing the parameters A, B $,\gamma,l,m$ and $\alpha$, the class $S_m^l[A,B,\alpha,\gamma]$ generalizes and extends other classes studied and introduced by various authors as Silverman
and Silvia[8] , Aouf and Cho[12] , and others . Note , that for A= - B = $l=1$ $m=0$ , and $\alpha_1=1$, the class $S_m^l[A,B,\alpha,\gamma]$ reduced to the class of $\alpha-$prestarlike functions $\alpha$, which was introducedby Sheil-Small et al [18].

Among several interesting definitions of fractional integrals given in the literature (cf.,e.g.,[20],[10] ,and [9] ), we find it be convenient , to recall here the following definition

**Definition 2:** For real numbers $\beta>,0$ $,\delta,$ and $\eta$, the fractional integral operator $I_{0,z}^{\beta,\delta,\eta}$ is defined by

(1.15)
$$I_{0,z}^{\beta,\delta,\eta}f(z)=\frac{z^{-\beta-\delta}}{\Gamma(\beta)}\int_0^z(z-t)^{\beta-1}{}_2F_1(\beta+\delta,-\eta:1-\frac{t}{z})f(t)dt.$$

for $\beta>0$ *and* $k>\max(0,\delta-\eta)-1$, where f(z) is an analytic function in a simply connected region of the z-plane containing the origin , and the multiplicity of $(z-t)^{\beta-1}$ is removed by requiring $\log(z-t)$, to be real when it is easy to observe that ,

(1.15)
$$I_{0,z}^{\beta,\delta,\eta}f(z)=D_z^{-\beta}f(z) \qquad , \quad (\beta>0) \quad .$$

Where $D_z^{-\beta}$ is the fractional integral operator considered by Owa [19] .

**Lemma 1:** [9, p.415, lemma 3]
If $\beta>0, and\ k>\delta-\eta-1$ then

(1.16)
$$I_{0,z}^{\beta,\delta,\eta}z^k=\frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\beta+\eta+1)}z^{k-\beta}$$

## 2. Coefficient estimates

Theorem 1 : A function f(z) defined by (1.2) belongs to the class $S_m^l(A,B\ ,\alpha,\gamma)$ if and only if

(2.1)
$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n}{(n-1)!}\le(A-B)(1-\alpha),\text{were}$$

(2.2) $$\phi\ (\mathrm{n}) = \frac{(\alpha_1)_{n-1}\ ...(\alpha_l)_{n-1}}{(\beta_1)_{n-1}...(\beta_m)_{n-1}}\ .$$

and $C(\gamma, n)$ is given by (1.11) . The result is sharp .

Proof . Assuming that (2.1) holds and $|z|$=1, then form (1.13) and (1.10) we have

$$\left| z\,(H_m^1[\alpha_1]\phi_\gamma(z))' - H_m^l[\alpha_1]\phi_\gamma(z) \right|$$

$$-\left|\{(B+(A-B)(1-\alpha))H_m^l[\alpha_1]\phi_\gamma(z)\} - B\,z(H_m^l[\alpha_1]\phi_\gamma(z))'\right|$$

$$= \left| \sum_{n=2}^{\infty} (n-1)\ C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n\ z^n \right| - \left|(A-B)(1-\alpha)Z\right.$$

$$\left. + \sum_{n=2}^{\infty}\{B(n-1)-(A-B)(1-\alpha)\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n z^n\ \right|$$

$$\le$$

$$\sum_{n=2}^{\infty}\{(1-B)(n-1)+(A-B)(1-\alpha)\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n - (A-B)(1-\alpha)$$

$$\le 0.$$

Hence by maximum modulus principle $f(z) \in S_m^l(A, B\ , \alpha, \gamma)$.

Conversely , assume that f(z) is in the class $S_m^l(A, B\ , \alpha, \gamma)$ . Then

(2.3) $$\left| \frac{\dfrac{z\,(H_m^l[\alpha_1]\phi_\gamma(z))'}{H_m^l[\alpha_1]\phi_\gamma(z)} - 1}{B+(A-B)(1-\alpha) - B\dfrac{z\,(H_m^l[\alpha_1]\phi_\gamma(z))'}{H_m^l[\alpha_1]\phi_\gamma(z)}} \right| < 1 \qquad , z \in U\ .$$

$$= \frac{\left|\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n\, z^n\right|}{\left|(A-B)(1-\alpha)z + \sum_{n=2}^{\infty}\{B(n-1)-(A-B)(1-\alpha)\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n z^n\right|}$$

Since $|\mathrm{Re}(z)| \le |z|$ for any z , we fined from (2.3) that

(2.4) $$\operatorname{Re}\left\{\frac{\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\dfrac{\phi(n)}{(n-1)!}a_n z^n}{(A-B)(1-\alpha)\,z+\sum_{n=2}^{\infty}\{B(n-1)-(A-B)(1-\alpha)\}C(\gamma,n)\dfrac{\phi(n)}{(n-1)!}a_n z^n}<1\right\}$$

Now choosing , the value of z on the real axis so that $\frac{z\,(H_m^l[\alpha_1]\phi_\gamma(z))'}{H_m^l[\alpha_1]\phi_\gamma(z)}$ is real , then upon clearing the denominator in (2.4) and letting $z\to 1$ through real values we have

$$\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n \le (A-B)(1-\alpha)$$

$$+\sum_{n=2}^{\infty}\{B(n-1)-(A-B)(1-\alpha)\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}a_n$$

Which gives the desired assertion (2.1) .

Finally , we note that equality in (2.1) holds for the function

(2.5) $$f(z)=z-\frac{(A-B)(1-\alpha)(n-1)!}{\{(1-B)(n-1)+(A-B)(1-\alpha)\}C(\gamma,n)\phi(n)\ a_n}z^n \ .$$

## 3. Characterization Properties

**Theorem 2.** Let $a_i > 0$ (i = 1,2,3, …,p) and $b_j > 0$ (j =1,2,3,…,q) such that $\prod_{j=1}^{q} b_j \ge \prod_{i=1}^{p} a_i$ .Also let the function f( z) defined by (1.2) satisfy

(3.1) $$\sum_{n=1}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n}{(1-\alpha)(A-B)(n-1)!}\le\frac{\prod_{j=1}^{q}b_j}{\prod_{i=1}^{p}a_i}$$

for -1< $\alpha < 1$ , then

$H_q^p[\alpha_1]\,f(z)\in S_m^l[A,B,\alpha,\gamma]$ where $\phi(n)$ is given by (2.2).

**Proof** . From (1.10)

$$H_q^p[\alpha_1]f(z)=H_q^p(\alpha_1,...,\alpha_p;b_1,...,b_q)\ f(z)=z-\sum_{n=2}^{\infty}\delta(n)\,a_n z^n$$

Where

(3.3) $$\delta(n)=\frac{(a_1)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}...(b_q)_{n-1}\quad (n-1)!} \qquad ,(n\geq 2)$$

Under the conditions stated in the theorem , we observe that the function $\delta(n)$ is non- increasing , that is ,it satisfies the Inequality

$\delta(n+1)\leq\delta(n)$ for all $n\geq 2$ ,and thus we

(3.4) $$0<\delta(n)\leq\delta(2)=\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}$$

Therefore , (3.1) and (3.4) yield

(3.5) $$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n\delta(n)}{(1-\alpha)(A-B)(n-1)!}$$

$$\leq\delta(2)\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n}{(1-\alpha)(A-B)(n-1)!}$$

$$\leq 1$$

Hence by Theorem 1 , we conclude that

$$H_p^q[\alpha_1]f(z)\in S_m^l[A,B,\alpha,\gamma].$$

**Remark** 1. The equality in (3.1) is attained for the function f(z) defined by

$$f(z)=z-\frac{(A-B)(1-\alpha)\prod_{j=1}^{m}B_j\prod_{j=1}^{q}b_j}{(1-B)+(A-B)(1-\alpha)C(\gamma,2)\prod_{i=1}^{l}\alpha_i\prod_{i=1}^{p}a_i}z^2$$

**Corollary 2** . Let $\lambda,\mu,\eta\in R$ such that

(3.7) $$\lambda\geq 0,\mu<2,\max(\lambda,\mu)-2<\eta\leq\frac{\lambda(\mu-3)}{\mu}$$

Also , let the function f(z) defined by (1.2) satisfy

(3.8) $$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n}{(A-B)(1-\alpha)(n-1)!}\leq\frac{(2-\mu)(2+\eta-\lambda)}{2(2+\eta-\mu)}$$

for $-1\leq\alpha<1,B>0$ , then $J_{0,z}^{\lambda,\mu,\eta}f(z)\in\beta-S^*(\alpha)$.

Proof .form Theorem 2 , by setting p = 3 ,q = 2 , $a_1=1,a_2=2,a_3=2+\eta-\mu$ , $b_1=2-\mu$ ,$b_2=2+\eta-\lambda$ and l =1 , m=0, $\alpha_1=1$ .

**Corollary 3**. Under the conditions stated in (3.7) , let the function f(z) defined by (1.2) satisfy

(3.9) $$\sum_{n=2}^{\infty}\frac{[n(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)a_n}{(1-\alpha)(A-B)(n-1)!}\leq\frac{(2-\mu)(2+\eta-\lambda)}{2(2+\eta-\mu)}$$

for $-1<\alpha<1, B\geq 0$ , *then* , $J_{0,Z}^{\lambda,\mu,\eta}f(z)\in B-k^{*}(\alpha)$.

**Proof.** The corollary follow from Theorem 2 by setting p=3 ,q=2 , $a_1=1, a_2=2, a_3=2+\eta-\mu$, $b_1=2-\mu$, $b_2=2+\eta-\lambda$ and l=1 , m=0 , $\alpha_1=2$.

## 4.Results Involving Modified Hadamared product.

**Theorem 3**. Let $\alpha_i>0, (i=1,2,...,l)$ and $B_j>0$ , $(j=1,2,3,...m)$ such that $\prod_{j=1}^{M}B_j\geq\prod_{i=1}^{l}\alpha_i$ .Also for the functions $f_j(z)$ $(j=1,2)$ defined by (1.3) ,let $f_1(z)\in S_m^l[A,B,\alpha,\xi]$ and $f_2(z)\in S[A,B,\alpha,\zeta]$ then $f_1 * f_2(z)\in S_m^l[A,\beta,\alpha,\delta]$

(4.1) $$\delta=1-\frac{(1-\xi)(1-\zeta)(A-B)\prod_{j=1}^{m}B_j}{(2+B-\xi)(2+B-\zeta)\prod_{i=1}^{l}\alpha_i-(1-\xi)(1-\zeta)(A-B)\prod_{j=1}^{m}B_j}$$

**Proof** . To prove the theorem it is sufficient to show that

(4.2) $$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\delta)]C(\gamma,2)\phi(n)a_{n,1}\ a_{n,2}}{(1-\delta)(A-B)(n-1)!}\leq 1$$

where $\phi(n)$ is defined in (2.2)and $\delta$ is defined in (4.1). Now by virtue of Cauchy-Schwarz inequality and Theorem 1 , it follow that

(4.3)

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\xi)]^{\frac{1}{2}}[(n-1)(1-B)+(A-B)(1-\zeta)]^{\frac{1}{2}}C(\gamma,n)\phi(n)\sqrt{a_{n,1}a_{n,2}}}{[(1-\xi)(1-\zeta)(A-B)]^{\frac{1}{2}}(n-1)!}$$

$\leq 1$

Hence (4.2)is true if

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\delta)]C(\gamma,2)\,\phi(n)\,a_{n,1}\ a_{n,2}}{(1-\delta)(A-B)(n-1)!}\leq$$

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\xi)]^{\frac{1}{2}}[(n-1)(1-B)+(A-B)(1-\zeta)]^{\frac{1}{2}}\ C(\gamma,2)\,\phi(n)\sqrt{a_{n,1}a_{n,2}}}{[(1-\xi)(1-\zeta)(A-B)]^{\frac{1}{2}}(n-1)!}$$

$$\leq 1$$

or equivalently

$$\sqrt{a_{n,1}a_{n,2}}\leq\{(n-1)(1-B)+(A-B)(1-\xi)\}^{\frac{1}{2}}\{(n-1)(1-B)+(A-B)(1-\zeta)\}^{\frac{1}{2}}$$

(4.4)

$$\times\frac{(1-\delta)(A-B)}{(n-1)(1-B)+(A-B)(1-\delta)}.$$

By virtue of (4.3) ,(4.2) is true if

$$\frac{\sqrt{(1-\xi)(1-\zeta)(A-B)}\quad(n-1)!}{(n-1)(1-B)+(A-B)(1-\xi)\}^{\frac{1}{2}}\{(n-1)(1-B)+(A-B)(1-\zeta)\}^{\frac{1}{2}}\phi(n)C(\gamma,2)}$$

$$\leq\{(n-1)(1-B)+(A-B)(1-\xi)\}^{\frac{1}{2}}\{(n-1)(1-B)+(A-B)(1-\zeta)\}^{\frac{1}{2}}\}$$

$$\times\frac{(1-\delta)(A-B)}{(n-1)(1-B)+(A-B)(1-\delta)}.$$

Which yields

$$\delta\geq 1-\frac{(1-n)(1-B)(1-\xi)(1-\zeta)(n-1)!}{\{(1-B)(n-1)+(A-B)(1-\xi)\}\{(1-B)(n-1)+(A-B)(1-\zeta)\}\phi(n)C(\gamma,2)-(1-\xi)(A-B)(1-\zeta)(n-1)!}$$

Under the stated conditions in the theorem , we observe that the function $\phi(n)$ is a decreasing for n $(n\geq 2)$ and thus (4.5) is satisfied if $\delta$ is given by (4.1) .Finally the result is sharp for

$$f_1(z)=z-\frac{(A-B)(1-\xi)}{[(1-B)+(A-B)(1-\xi)C(\gamma,2)}\ \frac{\prod_{j=1}^{m}B_j}{\prod_{i=1}^{l}\alpha_i}\ Z^2$$

$$f_2(z)=z-\frac{(A-B)(1-\zeta)}{[(1-B)+(A-B)(1-\zeta)C(\gamma,2)}\ \frac{\prod_{j=1}^{m}B_j}{\prod_{i=1}^{l}\alpha_I}\ Z^2 \quad .$$

**Theorem 4.**
Under the conditions stated in theorem 3 , let the functions $f_j(z)$ ( j = 1**, 2)** defined by (1.3) ,be in the class $S_m^l[A,B,\alpha,\gamma]$ then $f_1 * f_2(z)\in S_m^l[A,\beta,\alpha,\gamma]$ where

(4.6) $$\delta=1-\frac{(1-\alpha)^2(1-B)\ \ (A-B)\prod_{j=1}^{m}B_j}{(2+B-\alpha)^2(\prod_{i=1}^{l}\alpha_i)-\ \ (1-\alpha)^2\ (A-B)(\prod_{j=1}^{m}B_j)}$$

**Proof .** The result follows by setting $\alpha=\xi$ .

**Theorem 5 .**Under the conditions stated in Theorem 3 , let the functions $f_j(z)$ ( j =1 ,2 ) defined by (1.3) , be in the class $S_m^l[A,\beta,\alpha,\gamma]$ . Then

(4.7) $$h(z)=z-\sum_{n=1}^{\infty}(a_{a,1}^2+a_{n,2}^2)\ z^2$$

is in the class $S_m^l[A,\beta,\alpha,\gamma]$ , where

(4.8) $$\delta=1-\frac{2(1-\alpha)^2(1-B)\ \ (A-B)\prod_{j=1}^{m}B_j}{(2+B-\alpha)^2(\prod_{i=1}^{l}\alpha_i)-\ 2\ (1-\alpha)^2\ (A-B)(\prod_{j=1}^{m}B_j)}$$

**Proof .**In view of Theorem 1 , it is sufficient to prove that

(4.9) $$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)\ \ (a_{n,1}{}^2+a_{n,2}^2)}{(1-\delta)(A-B)(n-1)!}\le 1$$

Where $\phi(n)$ is defined in (2.2) and $\delta$ is defined in (4.8) as $f_j(z)\in S_m^l[A,\beta,\alpha,\gamma]$ ( j = 1 ,2) , Theorem 1 , yields

$$\sum_{n=2}^{\infty}\left[\frac{(n-1)(1-B)+(A-B)(1-\alpha)C(\gamma,n)\phi(n)}{(1-\alpha)(A-B)(n-1)!}\right]^2 \quad a_{n,2}^2$$

$$\leq \sum_{n=2}^{\infty}\left[\frac{(n-1)(1-B)+(A-B)(1-\alpha)\ C(\gamma,n)\phi(n)\quad a_{n,j}}{(1-\alpha)(A-B)(n-1)!}\right]^2 \leq 1$$

hence

(4.10) $$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(n-1)(1-B)+(A-B)(1-\alpha)\ C(\gamma,n)\phi(n)}{(1-\alpha)(A-B)(n-1)!}\right]^2 (a_{n,1}^2+a_{n,2}^2\ )\leq 1$$

(4.9) is it true if

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\delta)]C(\gamma,n)\phi(n)\ \ (\ a_{n,1}{}^2\ +a_{n,2}^2\ )}{(1-\delta)(A-B)(n-1)!}$$

$$\leq \sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(n-1)(1-B)+(A-B)(1-\alpha)\ C(\gamma,n)\phi(n)}{(1-\alpha)(A-B)(n-1)!}\right]^2 (a_{n,1}^2+a_{n,2}^2\ )$$

That is ,if

(4.11)

$$\delta=1-\frac{2(n-2)(1-\alpha)^2(1-B)\quad (A-B)(n-1)!}{[(n-1)\ \ (1-B)-\ \ (1-\alpha)(A-B)]^2\phi(n)C(\gamma,n)-2(1-\alpha)^2(A-B)(n-1)!}$$

Under the stated conditions in the theorem , we observe that the function $\phi(n)$ is a decreasing for n $(n\geq 2)$ ,
and thus (4.11)is satisfied if $\delta$ is given by (4.8)

## 5. Extreme points of the class $S_m^l[A,\beta,\alpha,\gamma]$

**Theorem 6** . Let $f_1(z)=z$ and

(5.1)

$$f_n(z) = z - \frac{(A-B)(1-\alpha)(n-1)!}{\{(n-1)(1-B)+(A-B)(1-\alpha)\}\phi(n)C(\gamma,n)} \; z^2 \qquad ,(n,2)$$

Then $f\ (z) \in S_m^l[A,\beta,\alpha,\gamma]$ if and only if it can be expressed in the form

(5.2) $$f(z) = \lambda_1\ f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n \ ,$$

Where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ and $\phi(n)$ is given in (2.2) .

**Proof** . Let (5.2) hold , then by (5.1) we have

$$f(z) = z - \sum_{]n=2}^{\infty} \frac{(A-B)(1-\alpha)(n-1)! \qquad \lambda_n\ z^n}{[(n-1)(1-B)+(A-B)(1-\alpha)]\phi(n)C(\gamma,n)}$$

Now

$$\sum_{n=2}^{\infty} (n-1)(1-B)+(A-B)(1-\alpha)\frac{\phi(n)C(\gamma,n)}{(n-1)!}a_n$$

$$= \sum_{n=2}^{\infty} (n-1)(1-B)+(A-B)(1-\alpha)\frac{\phi(n)C(\gamma,n)}{(n-1)!}a_n$$

$$\times \qquad \frac{(A-B)(1-\alpha)(n-1)! \qquad \lambda_n}{[(n-1)(1-B)+(A-B)(1-\alpha)]\phi(n)C(\gamma,n)}$$

$$= (1-\alpha)(A-B)\sum_{N=2}^{\infty}\lambda_n \leq (1-\alpha)(A-B)\sum_{n=1}^{\infty}\lambda_n$$

$$\leq (1-\alpha)(A-B)$$

Hence by Theorem 1 , $f\ (z) \in S_m^l[A,\beta,\alpha,\gamma]$

Conversely , suppose $f\ (z) \in S_m^l[A,\beta,\alpha,\gamma]$. Since

$$a_n \leq \frac{(A-B)(1-\alpha)(n-1)!}{[(n-1)(1-B)+(A-B)(1-\alpha)]\phi(n)C(\gamma,n)} \qquad (n \geq 2)$$

Setting

$$\lambda_n = \frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(1-\alpha)(A-B)(n-1)!}a_n$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n$ we get(5.2) .This is complete the proof of theorem .

## 6. Closure properties

**Theorem 7.** Let the function $f_j(z)$ defined by (1.3) be in the class $S_m^l[A,\beta,\alpha,\gamma]$. Then the function h(z) defined by

$$h(z)=z-\sum_{n=2}^{\infty} d_n z^n$$

belongs to $S_m^l[A,\beta,\alpha,\gamma]$ , where

$$d_n=\frac{1}{m}\sum_{j=1}^{m} a_{n,j} \qquad (a_{n,j}\ge 0)$$

**Proof .**Since $f_j(z)\in S_m^l[A,\beta,\alpha,\gamma]$ , it follow from Theorem 1 that

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(n-1)!}a_{n,j}\le (A-B)(1-\alpha) \tag{6.1}$$

Where $\phi(n)$ is given by (2.2) . therefore

$$\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(n-1)!}d_n$$

$$=\sum_{n=2}^{\infty}\frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(n-1)!}\frac{1}{m}\sum_{j=1}^{\infty}a_{n,j}$$

$$\le (A-B)(1-\alpha).$$

by theorem(6.1) , which show that $S_m^l[A,\beta,\alpha,\gamma]$ .

## 7. Integral transforms

Recently , Jung , Kim and Srivastava [11] introduced the following one-paramete family of integral operators

$$I^{\sigma}f(z)=\frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z(\log\frac{z}{t})^{\sigma-1}f(t)dt \qquad , \quad (\sigma>0) \ . \tag{7.1}$$

**Theorem 8 .**Let the function f(z) defined by (1.2) be the class $S_m^l[A,\beta,\alpha,\gamma]$ . Then the integral transform (7.1) belongs to $S_m^l[A,\beta,\alpha,\gamma]$ .

**Proof.** Using (1.2) and (7.1) we get

$$I^{\sigma}f(z)=z-\sum_{n=2}^{\infty}\left(\frac{2}{1+n}\right)^{\sigma}a_n z^n$$

Therefore

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(n-1)!} \left(\frac{2}{1-n}\right)^{\sigma} a_n$$

$$\leq \sum_{n=2}^{\infty} \frac{[(n-1)(1-B)+(A-B)(1-\alpha)]C(\gamma,n)\phi(n)}{(n-1)!} \quad a_n$$

$$\leq (A-B)(1-\alpha)$$

Which implies that $S_m^l[A,\beta,\alpha,\gamma]$ .

## References

[1] A.Schild ,On starlike functions of order $\alpha$ , Amer . J . Math .87 (1965) , 65-70 .

[2] A . W. Goodman , On uniformly convex functions , Ann . Polon . Math . 56(1991) 87- 92 .

[3] B.C. Carlosn and D.B.Shaffer . Starlike and pre-starlike hypergeometric functions SIAM J . Math . Anal . 15(1984) , 737-745 .

[4] D. Minda and W . Ma , Uniformly convex functions , Ann . Polon . Math . 57(1992) , 165-175.

[5] F . RØnning , Uniformly convex functions and a corresponding class of starlike functions , Proc . Amer . Math Soc. 118(1993) , 189-196 .

[6] G.Mur ., T .Rosy and M. Darus , A subclass of uniformly convex functions associated with certain fractional calculus operators , J . Ineq . Pure and Appl . Math., 6 (3) Art ., 86(2005) , 1-10 .

[7] H. Silverman , Univalent functions with negative coefficients , Proe Amer . Math . Soc .51 (1975) , 109-116.

[8] H. Silverman and E. M. Silvia, Prestarlike functions with negative coefficients, Internet. J.Math. Sci. 2(1979) , 427-439 .

[9] H. M . Srivestava, M. Saigo and S. Owa, A class of distortion involving certain operators of fractional calculus. J. Math. Anal. Appl. 131(1988),412-420 .

[10 ] H. M. Srivestava and S. Owa , Current topics in analytic function theory , world scientific publishing co. Singapore, New Jersey, London, and Hong Kong , 1992 .

[11] I . B . Jung , Y . C . Kim and H . M Srivastava , The Hardly space of analytic functions associated with certain one-parameter families of integral transform Math . Anal . Appl. , 179 (1993) , 138-147 .

[12] J. Dziok and H. M . Srivastava , Certain subclass f analytic functions associated with the generalized hypergeometric function , Integral Transform Spec . funct ., 14(2003), 7-18 .

[13] J .M shenan , on asubclass of $\beta$. Unifromly convex function. Int. Journal of Math. Analysis, vol.2(2008) 11, 499-512 .

[14] M. S. Robertson , On the theory of univalent functions , Ann. Of Math. 37(1936), 374-408 .

[15] R.K Raina and M .Saigo , A note on fractional calculus operators involving Fox' H- function on space $F_{p,\mu}$ , Recent Advances in Fractional Calculus , Global Publishing Co . Sauk Rapids , 1993, pp. 219-229 .

[16] R . K. Raina and T . S Nahar , Characterization properties for starlikeness and convexity of some subclasses of analytic functions involving a class of fractional derivative operator . Acta Math Univ . Comenianae . Vol. LXIX , 1(2000) , 1- 8 .

[17] St . Ruscheweyh , New criteria for univalent functions , Proc . Amer . Math . Soc 49(1975) , 109-115.

[18 ] T. Sheil-Small, H. Silverman and E. M. Convolution multipliers and starlike functions .J Analysis Math. 41(1982), 182-192.

[19]S. Owa, On the distortion Theorems . I, Kyungpook Math. J. 18(1978), 53-59 .

[20 ] S. Owa, M. Saigo and H. M. Srivestava , Some characterization Theorems for starlike and convex functions involving a certain fractional operator , J. Math. Anal. Appl. 140(1989),419-426.

[21] S. Owa and H . M . Srivastava , Univalent and starlike generalized hypergeometric functions , Canad . J . Math . 39 (5) (1987) , 1057-1077 .

[22] Yu . E . Hohlov , Operators and operations in the class of Univalent functions , Izv . Vyss. Ueebn . Zaved . Mat ., 10(1978) , 83-89 .

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