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Some Results on Fixed Points of Mappings in a 2-Metric Space

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Abstract

Here we have obtained some fixed pointic results for a class of contractive type mappings in a setting of 2-metric space.

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1 Introduction

The introduction of a 2-metric space was initially introduced by Gähler in a series of papers ([1]-[2]) in 1963-1965. Then about a decade after, Iseki[3] found some basic fixed point results in a setting of 2-metric space. After that some important fixed pointic results are obtained by Rhoades[5], Miczko et.al[4], Saha et al.[6] in this space. In the present paper we deal with the mixed type of contraction mappings[7] and also have found some interesting results in 2-metric space, where in each cases the idea of convergence of sum of a finite or infinite series of real constsnts plays a crucial role in the proof of fixed point theorems.

2 Preliminaries

Definition 2.1 . *Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if*
(i) given distinct elements x,y of X,there exists an element z of X such that $d(x, y, z) \neq 0$
(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal,
(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X, and
(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.
when d is a 2-metric on X, then the ordered pair (X, d) is called a 2-metric space.

Definition 2.2 . *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $a \in X$, $\lim d(x_n, x_m, a) = 0$ as $n, m \to \infty$.*

Definition 2.3 . *A sequence $\{x_n\}$ in X is convergent to an element $x \in X$ if for each $a \in X$, $\lim\limits_{n\to\infty} d(x_n, x, a) = 0$*

Definition 2.4 . *A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.*

3 Main results

Theorem 3.1 . *Let X be a complete 2-metric space. Let $0 \leq \beta_i, \gamma_i < 1$, (i=1,2,.......). Let T be a self map on X satisfying*

$$d(T^i(x), T^i(y), a) \leq \beta_i[d(x, T(x), a) + d(y, T(y), a] + \gamma_i d(x, y, a) \tag{1}$$

where $x, y, a \in X$, i=1,2,.......
Then T has a unique fixed point if $\sum\limits_{i=1}^{\infty}(\beta_i + \gamma_i) < \infty$

Proof For any $x \in X$, let $x_n = T^n(x)$ with $x = x_0$. Then

$$\begin{aligned} d(T(x_0), T^2(x_0), a) &= d(T(x_0), T(T(x_0)), a) \\ &\leq \beta_1 \left[d(x_0, T(x_0), a) + d(T(x_0), T^2(x_0), a\right] \\ &\quad +\gamma_1 d(x_0, T(x_0), a) \end{aligned}$$

$$\text{implies} \quad d(T(x_0), T^2(x_0), a) \leq \left(\frac{\beta_1 + \gamma_1}{1 - \beta_1}\right) d(x_0, T(x_0), a) \tag{2}$$

Now

$$\begin{aligned} d(x_n, x_{n+1}, a) &= d(T^n(x_0), T^{n+1}(x_0), a) \\ &= d(T^n(x_0), T^n(T(x_0)), a) \\ &\leq \beta_n[d(x_0, T(x_0), a) + d(T(x_0), T^2(x_0), a)] + \gamma_n d(x_0, T(x_0), a) \\ &\leq \beta_n \left[1 + \left(\frac{\beta_1 + \gamma_1}{1 - \beta_1}\right)\right] d(x_0, T(x_0), a) \\ &\quad + \gamma_n d(x_0, T(x_0), a) \qquad \text{by (2)} \end{aligned}$$

$$\text{implies} \quad d(x_n, x_{n+1}, a) \leq \left\{\left(\frac{1 + \gamma_1}{1 - \beta_1}\right)\beta_n + \gamma_n\right\} d(x_0, T(x_0), a) \tag{3}$$

Then

$$\begin{aligned} d(x_n, x_{n+2}, a) &= d(x_{n+2}, x_n, a) \\ &\leq d(x_{n+2}, x_n, x_{n+1}) + d(x_{n+2}, x_{n+1}, a) + d(x_{n+1}, x_n, a) \\ &= d(x_{n+2}, x_n, x_{n+1}) + \sum_{k=0}^{1} d(x_{n+k}, x_{n+k+1}, a) \end{aligned}$$

Similarly, $d(x_n, x_{n+3}, a) \leq \sum_{k=0}^{1} d(x_{n+3}, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{2} d(x_{n+k}, x_{n+k+1}, a)$
For any positive integer p,

$$d(x_{n+p}, x_n, a) \leq \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a) \tag{4}$$

Now

$$\begin{aligned} \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) &= d(x_{n+p}, x_n, x_{n+1}) + d(x_{n+p}, x_{n+1}, x_{n+2}) + \ldots\ldots \\ &\leq \left\{\left(\frac{1 + \gamma_1}{1 - \beta_1}\right)\beta_n + \gamma_n\right\} d(x_0, T(x_0), x_{n+p}) \\ &\quad + \left\{\left(\frac{1 + \gamma_1}{1 - \beta_1}\right)\beta_{n+1} + \gamma_{n+1}\right\} d(x_0, T(x_0), x_{n+p}) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{by (3)} \end{aligned}$$

$$\begin{aligned} \text{Also} \quad d(x_0, T(x_0), x_{n+p}) &= d(T(x_{n+p-1}), T(x_0), x_0) \\ &\leq \beta_1[d(x_{n+p-1}, x_{n+p}, x_0) + d(x_0, x_1, x_0)] \\ &\quad + \gamma_1 d(x_{n+p-1}, x_0, x_0) \\ &= \beta_1 d(x_{n+p-1}, x_{n+p}, x_0) \end{aligned}$$

Let us put $n+p-1=m$, then

$$\begin{aligned} d(x_0,T(x_0),x_{n+p}) &\leq \beta_1 d(x_m,x_{m+1},x_0) \\ &\leq \beta_1\left\{\left(\frac{1+\gamma_1}{1-\beta_1}\right)\beta_m+\gamma_m\right\}d(x_0,x_1,x_0) \qquad \text{by (3)} \\ &= 0 \end{aligned}$$

which implies $\sum_{k=0}^{p-2} d(x_{n+p},x_{n+k},x_{n+k+1})=0$. Then from (4)

$$\begin{aligned} d(x_{n+p},x_n,a) &\leq \sum_{k=0}^{p-1} d(x_{n+k},x_{n+k+1},a) \\ &\leq \sum_{k=0}^{p-1}\left\{\left(\frac{1+\gamma_1}{1-\beta_1}\right)\beta_{n+k}+\gamma_{n+k}\right\}d(x_0,T(x_0),a) \qquad \text{by (3)} \\ &= \left\{\left(\frac{1+\gamma_1}{1-\beta_1}\right)\sum_{k=0}^{p-1}\beta_{n+k}+\sum_{k=0}^{p-1}\gamma_{n+k}\right\}d(x_0,T(x_0),a) \end{aligned}$$

Now since $\sum_n(\beta_n+\gamma_n)<\infty$, $d(x_{n+p},x_n,a)\to 0$ as $n\to\infty$. So $\{x_n\}$ is a cauchy sequence in X and by completeness of X, x_n converges to a point $u\in X$. Again

$$\begin{aligned} d(x_{n+1},T(u),a) &= d(T^{n+1}(x_0),T(u),a) \\ &= d(T(T^n(x_0)),T(u),a) \\ &\leq \beta_1[d(T^n(x_0),T^{n+1}(x_0),a)+d(u,T(u),a)] \\ &\quad +\gamma_1 d(T^n(x_0),u,a) \\ &= \beta_1[d(x_n,x_{n+1},a)+d(u,T(u),a)]+\gamma_1 d(x_n,u,a) \end{aligned}$$

Taking limit on bothsides as $n\to\infty$, we get $d(u,T(u),a)\leq\beta_1 d(u,T(u),a)$ implies $T(u)=u$.
For uniqueness, let u, v be two fixed points of T.

$$\text{Then} \quad d(u,v,a)=d(T(u),T(v),a) \leq \beta_1[d(u,T(u),a)+d(v,T(v),a)] +\gamma_1 d(u,v,a)$$

gives $d(u,v,a)\leq\gamma_1 d(u,v,a)\Rightarrow u=v$ as $0\leq\gamma_1<1$.

Theorem 3.2 *Let X be a 2-metric space. Let* $0\leq\beta_i,\gamma_i<1$ *(i=1,2,.......) with* $\sum_n(\beta_n+\gamma_n)<\infty$. *Let T be a self map on X satisfying (1):*

$$d(T^i(x),T^i(y),a)\leq\beta_i[d(x,T(x),a)+d(y,T(y),a]+\gamma_i d(x,y,a)$$

where $x,y,a\in X$; *i=1,2,........ If for some* $x\in X$, $\{T^n(x)\}$ *has a subsequence* $\{T^{n_k}(x)\}$ *with* $\lim_k\{T^{n_k}(x)\}=u\in X$. *Then u is the unique fixed point of T.*

Proof. We have for $x, a \in X$,

$$\begin{aligned} d(u,T(u),a) &\leq d(u,T(u),T^{n_k+1}(x)) + d(u,T^{n_k+1}(x),a) \\ &\quad + d(T^{n_k+1}(x),T(u),a) \end{aligned} \tag{5}$$

$$\begin{aligned} \text{Now}\quad d(T^{n_k+1}(x),T(u),a) &= d(T(T^{n_k}(x)),T(u),a) \\ &\leq \beta_1[d(T^{n_k}(x),T^{n_k+1}(x),a) + d(u,T(u),a)] \\ &\quad + \gamma_1 d(T^{n_k}(x),u,a) \end{aligned}$$

Then from (5),

$$\begin{aligned} d(u,T(u),a) &\leq d(u,T(u),T^{n_k+1}(x)) + d(u,T^{n_k+1}(x),a) \\ &\quad + \beta_1[d(T^{n_k}(x),T^{n_k+1}(x),a) + d(u,T(u),a)] \\ &\quad + \gamma_1 d(T^{n_k}(x),u,a) \end{aligned}$$

Taking limit as $k \to \infty$ on bothsides of the inequality we get $d(u,T(u),a) \leq \beta_1 d(u,T(u),a)$ implies $d(u,T(u),a) = 0$. So $u = T(u)$ and uniqueness follows very immediate.

Theorem 3.3 *Let X be a complete 2-metric space and T is a self map on X satisfying*

$$d(T^i(x),T^i(y),a) \leq \beta_i[d(x,T(y),a) + d(y,T(x),a] + \gamma_i d(x,y,a) \tag{6}$$

for all $x, y, a \in X$ *with* $0 \leq \beta_i, \gamma_i < 1$ *for i=1,2,....... and* $\sum_{i=1}^{\infty}(\beta_i + \gamma_i) < \infty$ *Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and $x_n = T^n(x_0)$; n=1,2,.....,with $x_0 = T^0(x_0)$ Then by (6)

$$\begin{aligned} d(T(x_0),T^2(x_0),a) &\leq \beta_1[d(x_0,T^2(x_0),a) + d(T(x_0),T(x_0),a)] \\ &\quad + \gamma_1 d(x_0,T(x_0),a) \\ &= \beta_1 d(x_0,T^2(x_0),a) + \gamma_1 d(x_0,T(x_0),a) \end{aligned} \tag{7}$$

Also we have by (iv) of definition 2.1

$$\begin{aligned} d(x_0,T(x_0),a) &\leq d(x_0,T(x_0),T^2(x_0)) + d(x_0,T^2(x_0),a) \\ &\quad + d(T^2(x_0),T(x_0),a) \end{aligned} \tag{8}$$

Then from (7) and (8),

$$\begin{aligned} d(T(x_0),T^2(x_0),a) &\leq \left(\frac{\beta_1+\gamma_1}{1-\gamma_1}\right) d(x_0,T^2(x_0),a) \\ &\quad + \left(\frac{\gamma_1}{1-\gamma_1}\right) d(x_0,T(x_0),T^2(x_0)) \end{aligned} \tag{9}$$

Again

$$\begin{aligned} d(x_0, T(x_0), T^2(x_0)) &= d(T(x_0), T^2(x_0), x_0) \\ &\le \beta_1 \left[d(x_0, T^2(x_0), x_0) + d(T(x_0), T(x_0), x_0) \right] \\ &\quad + \gamma_1 d(x_0, T(x_0), x_0) \end{aligned}$$

$$\text{implies} \quad d(x_0, T(x_0), T^2(x_0)) = 0 \tag{10}$$

Therefore from (9) and (10),

$$d(T(x_0), T^2(x_0), a) \le \left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1} \right) d(x_0, T^2(x_0), a) \tag{11}$$

Similarly,

$$\begin{aligned} d(x_n, x_{n+1}, a) &= d(T^n(x_0), T^n(T(x_0)), a) \\ &\le \beta_n \left[d(x_0, T^2(x_0), a) + d(T(x_0), T(x_0), a) \right] \\ &\quad + \gamma_n d(x_0, T(x_0), a) \\ &\le \beta_n d(x_0, T^2(x_0), a) + \gamma_n d(x_0, T(x_0), a) \\ &\le \beta_n d(x_0, T^2(x_0), a) + \gamma_n d(x_0, T(x_0), T^2(x_0)) \\ &\quad + \gamma_n d(x_0, T^2(x_0), a) + \gamma_n d(T^2(x_0), T(x_0), a) \\ &\qquad \text{by (iv) of definition 2.1} \\ &= (\beta_n + \gamma_n) d(x_0, T^2(x_0), a) + \gamma_n d(T^2(x_0), T(x_0), a) \\ &\qquad \text{by (10)} \\ &\le (\beta_n + \gamma_n) d(x_0, T^2(x_0), a) + \gamma_n \left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1} \right) d(x_0, T^2(x_0), a) \\ &\qquad \text{by (11)} \end{aligned}$$

Therefore

$$d(x_n, x_{n+1}, a) \le \left[(\beta_n + \gamma_n) + \gamma_n \left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1} \right) \right] d(x_0, T^2(x_0), a) \tag{12}$$

Then proceeding in the same way as Theorem 3.1, for any $p > 0$,
$d(x_n, x_{n+p}, a) \le \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)$

$$\begin{aligned} \text{Now} \quad d(x_0, T(x_0), x_{n+p}) &= d(T(x_{n+p-1}), T(x_0), x_0) \\ &\le \beta_1 [d(x_{n+p-1}, T(x_0), x_0) + d(x_0, x_{n+p}, x_0)] \\ &\quad + \gamma_1 d(x_{n+p-1}, x_0, x_0) \\ &= \beta_1 d(x_{n+p-1}, T(x_0), x_0) \end{aligned}$$

$$\begin{aligned} \text{So}\quad d(x_0, T(x_0), x_{n+p}) &\le \beta_1 d(x_{n+p-1}, T(x_0), x_0) \\ &\le \beta_1^2 d(x_{n+p-2}, T(x_0), x_0) \\ &\le \ldots\ldots\ldots\ldots\ldots\ldots \\ &\le \beta_1^{n+p} d(x_0, T(x_0), x_0) \end{aligned}$$

$$\text{implies}\quad d(x_0, T(x_0), x_{n+p}) = 0 \tag{13}$$

Therefore

$$\begin{aligned} \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) &= d(x_{n+p}, x_n, x_{n+1}) + d(x_{n+p}, x_{n+1}, x_{n+2}) + \ldots\ldots \\ &\le \left[(\beta_n + \gamma_n) + \gamma_n\left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1}\right)\right] d(x_0, T(x_0), x_{n+p}) \\ &\quad + [(\beta_{n+1} + \gamma_{n+1}) \\ &\quad + \gamma_{n+1}\left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1}\right)\Bigg] d(x_0, T(x_0), x_{n+p}) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{by (12)} \end{aligned}$$

Then by (13), $\sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = 0.$ Thus

$$\begin{aligned} d(x_n, x_{n+p}, a) &\le \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a) \\ &\le \sum_{k=0}^{p-1}\left[(\beta_{n+k} + \gamma_{n+k}) + \gamma_{n+k}\left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1}\right)\right] d(x_0, T^2(x_0), a) \\ &\qquad \text{by (12)} \\ &\le \left[\sum_{k=0}^{p-1}(\beta_{n+k} + \gamma_{n+k}) + \left(\frac{\beta_1 + \gamma_1}{1 - \gamma_1}\right)\sum_{k=0}^{p-1}\gamma_{n+k}\right] d(x_0, T^2(x_0), a) \\ &\to 0 \text{ as } n \to \infty, \quad \text{since } \sum_n (\beta_n + \gamma_n) < \infty. \end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence in X and by completeness of X, $\lim_n x_n = u$ (say); $u \in X$. Now

$$\begin{aligned} d(x_{n+1}, T(u), a) &= d(T(T^n(x_0)), T(u), a) \\ &\le \beta_1 [d(x_n, T(u), a) + d(u, x_{n+1}, a)] + \gamma_1 d(x_n, u, a) \end{aligned}$$

Taking limit as $n \to \infty$ on bothside we get $d(u, T(u), a) \le \beta_1 d(u, T(u), a)$ $\Rightarrow T(u) = u$. Therefore u is a fixed point of T and uniqueness of u is also very clear.

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