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Geometric Probabilities for an Arbitrary Convex Body of Revolution in $\mathbf{E}_3$ and Certain Lattices

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Abstract

In this note we solve problems of Buffon type for an arbitrary convex body of resolution $\mathbf{K}$ in the euclidean space $\mathbf{E}_3$ and a particular lattice $\mathcal{R}$. As particular case we study the probability of intersection between a random sphere and the sides of $\mathcal{R}$.

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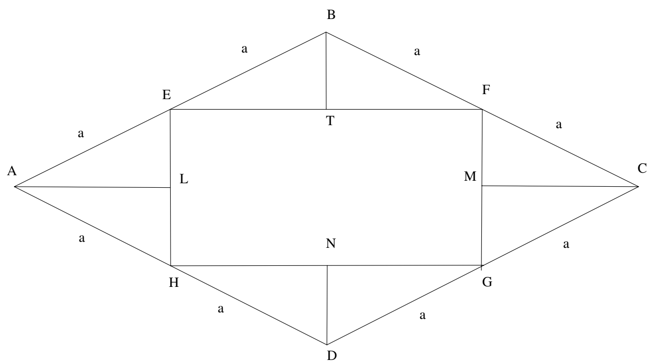
Problems of geometric probability for an arbitrary convex body of resolution in the euclidean space $\mathbf{E}_3$ has been investigated in [1]. In [9] Buffon's problem is solved for a lattice of right-angled parallelepipeds in the 3-dimensional space. In this note we want to use the results in [5] for to solve problems of intersection for a particular lattice that we describe: the fundamental cell $\mathcal{C}_0$ of the lattice $\mathcal{R}$ is a right-angled prism of height c and whose basis is the following:
Let $\mathbf{K}$ be an arbitrary convex body of resolution with centroid G and oriented axis of rotation r. The line r is determined by the angle ϑ between r and the z-axis and by the angle φ between the projection of r on the xy-plane and the x-axis. Hence $r = r(\vartheta, \varphi)$. Then the length $\mathcal{L}$ of the projection of $\mathbf{K}$ on the z-axis is given by

$$\mathcal{L}(\vartheta, \varphi) = p(\vartheta, \varphi) + p(\pi - \vartheta, \varphi)$$

where $p(\vartheta, \varphi)$ is the distance from G to the xy-plane when $\mathbf{K}$ is tangent to the xy-plane. Now let $\mathcal{C}_0$ be a fundamental cell of the lattice $\mathcal{R}$ and assume



$\mathcal{C}_{0,\pi}$: Basis of the prism of the fundamental cell $\mathcal{C}_0$.

that the two 3-dimensional random variables defined by the coordinates of G and by the triangle $(\vartheta, \varphi, \psi)$ are uniformly distributed in the cell $\mathcal{C}_0$ and in $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ respectively.

We denote by $\mathcal{M}_{\mathcal{C}_0}$ the set of all test bodies $\mathbf{K}$ whose centroid G lies in $\mathcal{C}_0$ and by $\mathcal{N}_{\mathcal{C}_0}$ the set of bodies $\mathbf{K}$ that are completely contained in $\mathcal{C}_0$.

We want to compute the probability $p_{\mathbf{K},\mathcal{R}}$ that the body $\mathbf{K}$ intersects the lattice $\mathcal{R}$. Denoting with μ the Lebesgue measure, the probability is given by

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{\mu(\mathcal{N}_{\mathcal{C}_0})}{\mu(\mathcal{M}_{\mathcal{C}_0})}. \tag{1}$$

Consider for fixed $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi]$ the set of points $P \in \mathcal{C}_0$ for with the body $\mathbf{K}$ with centroid P and rotation axis r does not intersect the boundary $\partial\mathcal{C}_0$ and let $\mathcal{C}(\vartheta, \varphi)$ the topological closure of this open subset of $\mathcal{C}_0$. We will assume that the body $\mathbf{K}$ is *small*[1] with respect to the lattice $\mathcal{R}$.

Denoting with $Diam(\mathbf{K})$ the diameter of the body $\mathbf{K}$, using the general result in [5], $\mathbf{K}$, is said small (respect to $\mathcal{R}$) iff

$$Diam(\mathbf{K}) < \Big(c, \frac{3\sqrt{3}}{8(3+\sqrt{3})}a\Big).$$

Using the kinematic measure (see:[9])

$$d\mathbf{K} = dx \wedge dy \wedge dz \wedge d\Omega \wedge d\psi, \tag{2}$$

[1]We say that the body $\mathbf{K}$ is small with respect to $\mathcal{R}$, if the polyhedrons sides of $\mathcal{C}(\vartheta, \varphi)$ and $\mathcal{C}_0$ are pairwise parallel.

where x, y, z are the coordinates of G, $d\Omega = \sin\vartheta d\vartheta \wedge d\varphi$, and ψ is angle of rotation about r. If $S \subseteq \mathbf{E}_3$ is a measurable subset we denote with $vol(S)$ the Euclidean volume of S. We have (see:[5])

$$\mu(\mathcal{M}_{\mathcal{C}_0}) = 8\pi^2 vol(\mathcal{C}_0), \tag{3}$$

$$\mu(\mathcal{N}_{\mathcal{C}_0}) = 2\pi \int_0^{2\pi} \left(\int_0^{\pi} vol(\mathcal{C}(\vartheta,\varphi)) \cdot \sin\vartheta d\vartheta \right) d\varphi. \tag{4}$$

Hence

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{1}{4\pi vol(\mathcal{C}_0)} \int_0^{2\pi} \left(\int_0^{\pi} vol(\mathcal{C}(\vartheta,\varphi)) \sin\vartheta d\vartheta \right) d\varphi. \tag{5}$$

Theorem 1. *If* $\mathbf{K}$ *is small with respect to* $\mathcal{R}$*, the probability* $p_{\mathbf{K},\mathcal{R}}$ *is given by*

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{\Lambda}{8\sqrt{3}\pi a^2 c}. \tag{6}$$

where:

$$\Lambda := \int_0^{2\pi} \Big(\int_0^{\pi} (a - \mathcal{L}(\vartheta_1(\vartheta,\varphi), \varphi_1(\vartheta,\varphi))) \cdot$$
$$(a\sqrt{3} - \mathcal{L}(\vartheta_2(\vartheta,\varphi), \varphi_2(\vartheta,\varphi))) \cdot (c - \mathcal{L}(\vartheta,\varphi)) +$$
$$\Big(1 - \frac{\mathcal{L}(\vartheta,\varphi)}{c}\Big) \cdot \Big(1 - \frac{2p_1}{a\sqrt{3}} - \frac{2p_2}{a} - \frac{4p_3}{a\sqrt{3}}\Big)^2 \cdot vol(\mathcal{C}_0^{(2)})\Big) \sin\vartheta d\vartheta \Big) d\varphi.$$

and

$$\vartheta_1(\vartheta,\varphi) := \arccos(\sin\vartheta\cos\varphi), \quad \varphi_1(\vartheta,\varphi) := \arctan\Big(\frac{\cot\vartheta}{\sin\varphi}\Big),$$

$$\vartheta_2(\vartheta,\varphi) := \arccos\Big(\sin\vartheta\sin\varphi\Big), \quad \varphi_2(\vartheta,\varphi) := \arctan\Big(\tan\vartheta\sin(\varphi+\pi/2)\Big),$$

$$\vartheta_3(\vartheta,\varphi) := \arccos(-\sin\vartheta\sin(\varphi+\pi/6)), \quad \varphi_3(\vartheta,\varphi) := \mathrm{arcctg}(-tan\vartheta\cos(\varphi_2+\pi/6)),$$

with

$$p_1 := p(\vartheta_1(\vartheta,\varphi), \varphi_1(\vartheta,\varphi)), \quad p_2 := p(\vartheta_2(\vartheta,\varphi), \varphi_2(\vartheta,\varphi)),$$
$$p_3 := p(\vartheta_3(\vartheta,\varphi), \varphi_3(\vartheta,\varphi)).$$

Proof: We use the following notations:

- $\mathcal{N}_1$ the set of all test bodies of revolution $\mathbf{K}$ completely contained in the prism whose base is the rectangle $EHGF$ (we say this prism $\mathcal{C}_0^{(1)}$);
- $\mathcal{N}_2$ the set of all test bodies of revolution $\mathbf{K}$ completely contained in the prism triangle whose base is the triangle ALH (we say this prism $\mathcal{C}_0^{(2)}$);

Consider for fixed $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi]$ the set of points $P \in \mathcal{C}_0$ for with the body $\mathbf{K}$ with centroid P and rotation axis r does not intersect the boundary $\partial\mathcal{C}_0^{(i)}$ $(i = 1, 2)$ and let $\mathcal{C}_0^{(i)}(\vartheta, \varphi)$ $(i = 1, 2)$ the topological closures of this open subsets of $\mathcal{C}_0^{(i)}$.

Then formula (1) becomes

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{\mu(\mathcal{N}_1) + 8\mu(\mathcal{N}_2)}{\mu(\mathcal{M}_{\mathcal{C}_0})}, \tag{7}$$

Now, let us consider the cell $\mathcal{C}_0^{(1)}$ with the coordinates $\vartheta_1, \varphi_1, \psi_1$. Follows [5] we put:

$$\vartheta_1^{(1)}(\vartheta_1, \varphi_1) := \arccos(\sin\vartheta_1 \cos\varphi_1),$$
$$\varphi_1^{(1)}(\vartheta_1, \varphi_1) := \arctan\Big(\frac{\text{cotg}\vartheta_1}{\sin\varphi_1}\Big),$$
$$\vartheta_2^{(1)}(\vartheta_1, \varphi_1) := \arccos\Big(\sin\vartheta_1 \sin\varphi_1\Big),$$
$$\varphi_2^{(1)}(\vartheta_1, \varphi_1) := arctan\Big(\tan\vartheta_1 \sin(\varphi_1 + \pi/2)\Big),$$

Hence we obtain the expression of the volume:

$$vol(\mathcal{C}_0^{(1)}(\vartheta_1, \varphi_1)) = (a - \mathcal{L}(\vartheta_1^{(1)}(\vartheta_1, \varphi_1), \varphi_1^{(1)}(\vartheta_1, \varphi_1)))\cdot$$
$$(a\sqrt{3} - \mathcal{L}(\vartheta_2^{(1)}(\vartheta_1, \varphi_1), \varphi_2^{(1)}(\vartheta_1, \varphi_1))) \cdot (c - \mathcal{L}(\vartheta_1, \varphi_1)),$$

Let us consider the cell $\mathcal{C}_0^{(2)}$ with the coordinates $\vartheta_2, \varphi_2, \psi_2$. We denote:

$$\vartheta_1^{(2)}(\vartheta_2, \varphi_2) := \arccos(\sin\vartheta_2 \cos\varphi_2),$$
$$\varphi_1^{(2)}(\vartheta_2, \varphi_2) := \arctan\Big(\frac{\text{cotg}\vartheta_2}{\sin\varphi_2}\Big),$$
$$\vartheta_2^{(2)}(\vartheta_2, \varphi_2) := \arccos\Big(\sin\vartheta_2 \sin\varphi_2\Big),$$

$$\varphi_2^{(2)}(\vartheta_2,\varphi_2) := \arctan\Big(\tan\vartheta_2\sin(\varphi_2+\pi/2)\Big),$$
$$\vartheta_3^{(2)}(\vartheta_2,\varphi_2) := \arccos(-\sin\vartheta_2\sin(\varphi_2+\pi/6)),$$
$$\varphi_3^{(2)}(\vartheta_2,\varphi_2) := \operatorname{arcctg}(-\tan\vartheta_2\cos(\varphi_2+\pi/6)).$$

Hence we give the volume of $\mathcal{C}_0^{(2)}$:

$$vol(\mathcal{C}_0^{(2)}(\vartheta_2,\varphi_2)) = \Big(1-\frac{\mathcal{L}(\vartheta_2,\varphi_2)}{c}\Big)\cdot \Big(1-\frac{2p_1}{a\sqrt{3}}-\frac{2p_2}{a}-\frac{4p_3}{a\sqrt{3}}\Big)^2\cdot vol(\mathcal{C}_0^{(2)}).$$

After a changing of variables $(\vartheta_i,\varphi_i,\psi_i)\to(\vartheta,\varphi,\psi)$, for $i=1,2$ we compute:

$$p_{\mathbf{K},\mathcal{R}} = 1-\frac{\mu(\mathcal{N}_1)+8\mu(\mathcal{N}_2)}{\mu(\mathcal{M}_{\mathcal{C}_0})} = \tag{8}$$
$$=1-\frac{1}{4\pi vol(\mathcal{C}_0)}\Big[\int_0^{2\pi}\left(\int_0^{\pi} vol(\mathcal{C}_0^{(1)}(\vartheta_1,\varphi_1))\sin\vartheta_1 d\vartheta_1\right)d\varphi_1+$$
$$+8\left(\int_0^{2\pi}(\int_0^{\pi} vol(\mathcal{C}_0^{(2)}(\vartheta_2,\varphi_2))\sin\vartheta_2 d\vartheta_2\right)d\varphi_2\Big].$$

With the following position:

$$\Lambda := \int_0^{2\pi}\Big(\int_0^{\pi}(a-\mathcal{L}(\vartheta_1(\vartheta,\varphi),\varphi_1(\vartheta,\varphi)))\cdot$$
$$(a\sqrt{3}-\mathcal{L}(\vartheta_2(\vartheta,\varphi),\varphi_2(\vartheta,\varphi)))\cdot(c-\mathcal{L}(\vartheta,\varphi))+$$
$$\Big(1-\frac{\mathcal{L}(\vartheta_2,\varphi_2)}{c}\Big)\cdot\Big(1-\frac{2p_1}{a\sqrt{3}}-\frac{2p_2}{a}-\frac{4p_3}{a\sqrt{3}}\Big)^2\cdot vol(\mathcal{C}_0^{(2)})\Big)\sin\vartheta d\vartheta\Big)d\varphi.$$

We obtain

$$p_{\mathbf{K},\mathcal{R}} = 1-\frac{\Lambda}{4\pi vol(\mathcal{C}_0)}. \tag{9}$$

□

As application of the theorem we can compute the probability of intersection with a side of the lattice $\mathcal{R}$ when $\mathbf{K}$ is a random sphere Σ of constant radius R and D as diameter.

Corollary 2. *If Σ is small with respect to $\mathcal{R}$, the probability $p_{\Sigma,\mathcal{R}}$ is given by*

$$p_{\Sigma,\mathcal{R}} = 1 - \frac{1}{2\sqrt{3}a^2c}\Big\{(a-D)(a\sqrt{3}-D)(c-D) + \tag{10}$$

$$8\Big[\frac{a\sqrt{3}}{2} - (3+\sqrt{3})R\Big]\Big[\frac{a}{2} - \Big(1+\sqrt{3}\Big)R\Big](c-R)\Big\}.$$

Remark 3. It is possible to obtain the result in Corollary 2 using theorems of E.Bosetto in [3].

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