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On the Expected Number of Economic Equilibria

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Abstract

The problem of multiplicity of equilibria in exchange economies is considered in the present paper from a probabilistic point of view. For a given family of exchange equilibria, considered as a set of all nonnegative linear combinations of a finite number of excess demand functions, the expected number of equilibria can be found by a geometric approach. The results may be used to provide upper bounds for the number of equilibria in several special cases of families of exchange equilibria.

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1 Introduction

One of the classical problems of mathematical economics is the determinateness of economic equilibria, investigating conditions such that there is a unique system of relative prices at which demand equals supply. It is by now well established that economies having a unique equilibrium constitute a very small subset of the economies which are of potential interest, and following Debreu [4], most work has been directed towards showing that almost all economies have only finitely many equilibria. For a detailed exposition of this theory, the reader is referred to advanced texts as e.g. Balasko [2], Green, Mas-Colell and Whinston [11].

In the present work, we propose another approach to the question of determinateness, finding the probability distribution of the number of equilibria, given some initial distribution of agents' characteristics. The tools used are based on work by Edelman and Kostlan [8]. The equilibrium property is transformed to one of geometry, namely to a situation where a vector of parameters characterizing the economic agents and the vector of their excess demand are orthogonal, and the probability of such an event may then by computed from the data of the problem.

The paper is organized as follows: In the next section, we introduce the basic formalism. The main result are to be found in section 3 for the simplest case of two commodities, and in section 4 with for the multi-commodity case.

2 Exchange economies and the number of equilibria

For the purpose of our discussion in the present section, an *exchange economy* is an array $\mathcal{E} = (u_i, \omega_i)_{i=1}^m$ of consumers (thus, we do not consider economies with production), where for each i,

- $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ is a utility function, defined on all nonnegative l-vectors (consumption of the l commoditites) and assumed to be continuous and monotonic in the sense that $x'_{ih} > x_{ih}$ for $h = 1, \ldots, l$ implies $u_i(x'_i) > u_i(x_i)$,
- $\omega_i \in \mathbb{R}^l_{++}$ is an initial endowment of consumer i.

Let $\triangle_{l-1} = \{x \in \mathbb{R}^l_+ \mid \sum_{h=1}^l x_h = 1\}$ be the standard simplex in $\mathbb{R}^l$. The demand of the consumer i with characteristics (u_i, ω_i) at the price (system) $p \in \operatorname{int} \triangle_{l-1}$ is

$$\xi_{(u_i,\omega_i)}(p) = \{x_i \mid p \cdot (x_i - \omega_i) = 0,\ [u_i(x'_i) > u_i(x_i)] \Rightarrow [p \cdot (x_i - \omega_i) > 0]\}.$$

We restrict our attention to cases where the set $\xi_{(u_i,\omega_i)}(p)$ is a singleton, so that $\xi_{(u_i,\omega_i)}$, or, shorthand ξ_i, is a well-behaved function.

Assumption 1. *The consumer* (u_i, ω_i) *is satisfies the smoothness condition if*
(i) u_i *is* C^2,
(ii) for each $x_i \in \mathbb{R}^l_{++}$, $Du_i(x_i) \in \mathbb{R}^l_{++}$,
(iii) for each $x_i \in \mathbb{R}^l_{++}$, *the restriction of the quadratic form* $D^2 u_i(x_i)$ *to* $\{x'_i \mid Du_i(x_i) \cdot x'_i = 0\}$ *is negative definite.*

If the consumer satisfies Assumption 1, demand is single-valued, and the excess demand function is C^1 (cf. e.g., Debreu [5], Balasko [2]). Define individual *excess demand* ζ_i by $\zeta_i(p) = \xi_i(p) - \omega_i$, for $p \in \operatorname{int} \triangle_{l-1}$, and aggregate excess demand ζ by $\zeta(p) = \sum_{i=1}^m \zeta_i(p)$. Then the excess demand correspondence satisfies the property

$$p \cdot \zeta(p) = 0$$

for all $p \in \operatorname{int} \triangle_{l-1}$, known as *Walras' law.*

Example 1. A consumer (u_i, ω_i) is said to be *homothetic* if the utility function u_i is positively homogeneous of some degree $s > 0$, so that $u_i(\lambda x_i) = \lambda^s u_i(x_i)$

for all $x_i \in \mathbb{R}^l_+$ and $\lambda > 0$. If (u_i, ω_i) is a homothetic consumer, then so is $(\mu u_i, \mu\omega_i)$ for any $\mu > 0$, and the associated demand function $\zeta_{(\mu u_i, \mu\omega_i)}$ satisfies

$$\zeta_{(\mu u_i, \mu\omega_i)}(p) = \mu\zeta_{(u_i,\omega_i)}(p)$$

for any $p \in \operatorname{int} \triangle_{l-1}$.

An *equilibrium price* (shorthand: an equilibrium) for $\mathcal{E}$ is a price vector p^0 such that

$$\zeta(p^0) = 0. \tag{1}$$

It is well-known that in the models of an economy as discussed here, equilibria exist (cf. e.g. the survey on existence theory in Debreu [7]). Here we are concerned with the possible multiplicity of equilibria, the fact that there may be more than one equilibrium price satisfying (1) above. It is easy to construct examples of excess demand functions with multiple zeros, and such counterexamples are in no way pathological. It was shown by Debreu [6] that any continuous function on an open subset of $\operatorname{int} \triangle_{l-1}$ satisfying Walras' law can be obtained as the aggregate excess demand function of an economy.

Example 2. In the special case of $l = 2$ (only two commodities), we have that $\triangle_1$ becomes the unit interval $[0, 1]$, to find an equilibrium we need only solve

$$\zeta_{11}(t) + \cdots + \zeta_{12}(t) = 0$$

for some $t^0 \in]0, 1[$, since by Walras' law we will also have

$$\zeta_{21}(t) + \zeta_{22}(t) = 0.$$

Geometrically, we consider the curve in $\mathbb{R}^2$ given by the rule $t \mapsto z(t) = (\zeta_{11}(t), \zeta_{21}(t))$ (excess demand for commodity 1 of each of the consumers). Clearly, t^0 is an equilibrium of this economy if $\zeta_{11}(t^0)+\zeta_{21}(t^0) = 0$, or, otherwise put,

$$e \cdot z(t) = 0.$$

The equilibrium conditions have a geometric interpretation: The diagonal vector should be orthogonal to the vector $z(t^0)$. We return to the geometric viewpoint in a later section.

Example 3. While uniqueness of equilibria does not hold in general, special properties of excess do lead to uniqueness. One such property is *gross substitution:* for all pairs of distinct commodities h and k,

$$\frac{\partial\zeta_h}{\partial p_k} > 0,$$

(increased price of commodity k entails increased excess demand for all other commodities). Another such property is *weak axiom of revealed preference:*, for all pairs (p^1, p^2) of price vectors,

$$p^1 \cdot \zeta(p^2) \leq 0 \text{ and } \zeta(p^1) \neq \zeta(p^2) \text{ implies } p^2 \cdot \zeta(p^1) > 0.$$

These assumptions are however restrictive (cf. Debreu's result [6] stated above), and they not easily reduced to properties of the individuals constituting the economy. For a survey of the conditions for uniqueness of equilibria, the reader is referred to Arrow and Hahn [1]. ○

3 Families of economies and the average number of equilibria

Following Hildenbrand [9] we consider economies as collections of consumers, sampled from a given set $\mathcal{C}^0$. Formally, an *economy* is a map a from $\mathcal{C}^0$ to $\mathbb{R}_+$, and a *family of economies* is a pair $(\mathcal{C}^0, \mu)$, where $\mathcal{C}^0$ is a finite set of consumers satisfying Assumption 1 (with cardinality $|\mathcal{C}^0| = n$), and μ is a probability distribution on $\mathbb{R}_+^{\mathcal{C}^0}$.

Intuitively, an economy is obtained by a suitable nonnegative number $a_1, \ldots, a_n$ of each of the consumers in $\mathcal{C}^0 = \{(u_1, \omega_1), \ldots, (u_n, \omega_n)\}$. However, in some applications, the a_i need not be natural numbers. The probability measure μ weighs the economies in the family according to their importance or relative frequency.

For $\mathcal{C}^0$ a finite set of consumers and $a \in \mathbb{R}^{\mathcal{C}^0}$, we define an *equilibrium* of a in the obvious way, that is as a price vector p^0 such that

$$\sum_{i=1}^{n} a_i \zeta_i(p^0) = 0,$$

and the set of equilibria of a as

$$W(a) = \{p^0 \in \triangle_{l-1} \mid \sum_{i=1}^{n} a_i \zeta_i(p^0) = 0\}.$$

The *number of equilibria* is the extended real-valued function $\nu : \operatorname{supp} \mu \to [0, +\infty]$ defined by

$$\nu(a) = |W(a)|,$$

and the *expected number of equilibria* (of the family $(\mathcal{C}^0, \mu)$ of economies) is

$$\mathrm{E}\,\nu = \begin{cases} \int \nu(a)\, d\mu(a) & \text{if } \nu \text{ is } \mu\text{-integrable,} \\ +\infty & \text{otherwise.} \end{cases}$$

In the following, we shall be concerned with assessing Eν for general families $(\mathcal{C}^0, \mu)$. Before we turn to this, some considerations of possible interpretations are in order, in particular since the notion of equilibrium was extended to situations not covered by that standard formulation of equilibrium theory.

Example 4. *(a)* Nonatomic economies with finitely many types (Hildenbrand [9], Hildenbrand and Kirman [10]): The probability distribution μ has support in $\{a \in \mathbb{R}_+^{\mathcal{C}^0} \mid \sum_{i=1}^n a_i = 1\}$. In the interpretation, the set of consumers are subsets of $[0,1]$ of length a_i. If ζ_i is the excess demand of consumer (u_i, ω_i), then aggregate excess demand is

$$\zeta = \sum_{i=1}^{n} a_i \zeta_i.$$

Thus, the family of economies may alternatively be considered as a probability distribution over sums of the individual excess demand functions.

(b) Densities ν with support in $\mathbb{R}_+^{\mathcal{C}^0}$: The finite counterpart of Example 4(a) is the situation where $\sum_{i=1}^n \zeta_i$ is interpreted as the excess demand of a finite economy with a_i agents of the type (u_i, ω_i) giving rise to excess demand function ζ_i. For this to make sense, we must have that each of the a_i is nonnegative integer, so that the support of ν must be restricted to $\mathbb{R}_+^{\mathcal{C}^0}$.

(c) Homothetic consumers: If ζ_i is an excess demand function of a homothetic consumer, then by the (first) aggregation theorem of Chipman [3], $a_i\zeta_i$ is the excess demand of the consumer having the same utility and endowment vector $a_i\omega_i$. Therefore, any family of economies over a finite set $\mathcal{C}^0_{hom}$ of homothetic consumers – even the family where the probability distribution has full support $\mathbb{R}_+^{\mathcal{C}^0}$ – has a straightforward interpretation, since each member of the support is a genuine exchange economy.

4 The average number of equilibria: The case of two commodities

We begin with the simple case of two commodities. Let $\mathcal{C}, \mu$ be a family of economies with $l = 2$. Then excess demand is a function of a single variable $t = p_1$, and by Walras' law it sufficies to consider the excess demand for commoditiy 1. The key observation in the study of $\mathsf{E}\,\nu$ is the following simple geometric version of the equilibrium condition, here formulated as a lemma:

Lemma 1. *Let $t^0 \in]0,1[$. Then t is a equilibrium for the random economy a if and only if $a = (a_1, \dots, a_n)$ is orthogonal to $(\zeta_{11}(t), \dots, \zeta_{n1}(t))$.* □

To proceed, we use a geometric argument, given in Edelman and Kostlan [8]: To find the average number of equilibria, we find for each t the set of economies (weighted by its density) for which t is an equilibrium price (using Lemma 1), and then integrate over t.

For ease of notation, we rename the curve in $\mathbb{R}^n$ of excess demands for the first commodity of each of the agents as $z(t) = (\zeta_{11}(t), \ldots, \zeta_{n1}(t))$, and we introduce the normalized version of this curve, $\gamma(t) = z(t)/\|z(t)\|$.

THEOREM 1. *The expected number of equilibria of the family* $(\mathcal{C}^0, \mu)$ *is given by*

$$\mathsf{E}\,\nu = \int_0^1 \left[\int_{\{\gamma(t)\}^\perp} |\gamma'(t) \cdot a| \, d\mu(a) \right] dt, \tag{2}$$

where $\{\gamma'(t)\}^\perp$ *denotes the subspace of* $\mathbb{R}^n$ *orthogonal to* $\gamma(t)$.

PROOF: Fix t and choose an orthonormal basis $e_1, \ldots, e_n$ such that $e_1 = \gamma(t)$ and $e_2 = \gamma'(t)/\|\gamma'(t)\|$. When we move from t to $t + dt$, the the hyperplane perpendicular to $\gamma(t)$ will sweep out a subset of $\mathbb{R}^n$. This set is the Cartesian product of a two-dimensional subset of span(e_1, e_2), having area $(\gamma'(t)dt)(|e_2 \cdot a|)$, with the entire span of the remaining $n - 2$ basis directions. The volume of the set is therefore

$$\|\gamma'(t)\| dt \int_{\mathbb{R}^{n-1}} |e_2 \cdot a| \, d\mu(a),$$

where the domain of integration is the $(n-1)$-dimensional space $\{\gamma'(t)\}^\perp$ orthogonal to e_1. Inserting, we get

$$\mathsf{E}\,\nu = \int_0^1 \left(\|\gamma'(t)\| \int_{\{\gamma(t)\}^\perp} \frac{|\gamma' t(a) \cdot a|}{\|\gamma'(t)\|} \, d\mu(a) \right) dt = \int_0^t \left[\int_{\{\gamma(t)\}^\perp} |\gamma'(t) \cdot a| \, d\mu(a) \right] dt,$$

which is (2). □

A simple application of Theorem 1 is given in the example below.

EXAMPLE 5. Consider the family $(\mathcal{C}^0, \mu)$ where $\mathcal{C}^0$ consists of 2 consumers having Cobb-Douglas utilities

$$u_i = x_1^{\alpha^i} x_2^{1-\alpha^i}, \ i = 1, 2,$$

and identical endowment $(\omega_1, \omega_2) = (1, 1)$, and where μ is the uniform distribution on $\{a \in \mathbb{R}^n_+ \mid a_1 + a_2 = 1\}$. It is well-known that economies in this family have a unique equilibrium, so the computation below serves mainly as an illustration of the method.

The excess demand function of (u_i, ω_i) is

$$z_i(t) = \zeta_{i1}(t) = \alpha^i \frac{t\omega_1^i + (1-t)\omega_2^i}{t} - \omega_1^i = \frac{\alpha^i}{t} - 1,$$

with $z_i'(t) = -\alpha^i t^{-2}$, $i = 1, 2$. It is seen that the set of a with $\gamma(t) \cdot a = z(t) \cdot a = 0$ and $a_1 + a_2 = 1$ is empty for $t \notin [\alpha^2, \alpha^2]$ (where both coordinates of $z(t)$ have the same sign), and uniquely determined as

$$a(t) = \left(\frac{t - \alpha^2}{\alpha^1 - \alpha^2}, \frac{\alpha^1 - t}{\alpha^1 - \alpha^2} \right)$$

for $t \in [\alpha^2, \alpha^1]$. Consequently, the integral in (2) reduces to

$$\int_{\alpha^2}^{\alpha^1} \frac{1}{\alpha^1 - \alpha^2} \, dt = 1$$

which gives us the expected result.

The example was sufficiently simple to allow for explicit computation of the expected number of equilibria using the formula of Theorem 1. In general it this is not possible, but the result may be used to derive upper bounds on the expected number of equilibria for given families of economies.

COROLLARY. *Let* $(\mathcal{C}^0, \mu)$ *be a family of economies with* $\operatorname{supp} \mu \subset \{a \in \mathbb{R}_+^{\mathcal{C}^0} \mid \sum_{i=1}^n a_i = 1\}$, *let* $t_{\min}$ *and* $t_{\max}$ *be such that* $t_{\min} \le t \le t_{\max}$ *for any* t *with* $\zeta_i(t) = 0$, $i \in \mathcal{C}'$, *and let*

$$M = \max_{t \in [t_{\min}, t_{\max}]} \max_i \frac{|\zeta_i'(t)|}{\sqrt{\zeta_1(t)^2 + \cdots + \zeta_n(t)^2}}.$$

Then $\mathsf{E}\nu \le M\sqrt{n}(t_{\max} - t_{\min})$.

PROOF: Using that

$$\gamma'(t) = \frac{d}{dt} \left(\frac{z(t)}{\|z(t)\|} \right) = \frac{z'(t)}{\|z(t)\|} + \gamma(t) \frac{d}{dt} \|z(t)\|,$$

and exploiting that in (2) we integrate over a such that $\gamma(t) \cdot a = 0$, we get

$$\mathsf{E}\,\nu = \int_{t_{\min}}^{t_{\max}} \left[\int_{\{\gamma(t)\}^\perp} \frac{|z'(t) \cdot a|}{\|z(t)\|} \, d\mu(a) \right] dt.$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \mathsf{E}\,\nu &\le \int_{t_{\min}}^{t_{\max}} \int_{\{\gamma(t)\}^\perp} \|a\| \frac{\|z'(t)\|}{\|z(t)\|} \, d\mu(a) \, dt \\ &\le \int_{t_{\min}}^{t_{\max}} \int_{\{\gamma(t)\}^\perp} \|a\| M\sqrt{n} \, d\mu(a) \, dt = M\sqrt{n}(t_{\max} - t_{\min}). \end{aligned}$$

$\square$

5 The general case of arbitrary number of commodities

Assume that the family of economies $(\mathcal{C}^0, \mu)$ with $\mathcal{C}^0 = \{(u_1, \omega_1), \ldots, (u_n, \omega_n)\}$ has l commodities, where $l < n$ for simplicity. We say that a price $p \in \triangle_{l-1}$ is a *commodity 1 equilibrium price* in the economy given by $(a_1, \ldots, a_n)$ if

$$\sum_{i=1}^{n} a_i \zeta_{i1}(p) = 0.$$

For $a \in \mathcal{C}^0$, the set of commodity 1 equilibria in a is denoted by $W^1(a)$, its cardinality by $\nu^1(a)$, and the expected number of commodity 1 equilibria in the family $(\mathcal{C}^0, \mu)$ by $\mathsf{E}\,\nu^1$. We use the notation $z_1(p) = (\zeta_{11}(p), \ldots, \zeta_{n1}(p))$ and $\gamma_1(p) = z_1(p)/\|z_1(p)\|$.

The following is essentially a restatement of Theorem 1 to deal with the situation, the new aspect being that the variable p is now $(l-1)$-dimensional.

LEMMA 2. *The expected number of commodity 1 equilibria of the family $(\mathcal{C}^0, \mu)$ satisfies*

$$\mathsf{E}\nu^1 \le \int_{\triangle_{l-1}} \|D_1 z_1(p)\| \cdots \|D_{l-1} z_1(p)\| \left[\int_{\{\gamma_1(p)\}^\perp} |\mathrm{proj}_{Dz_1(p)} a| \, d\mu(a) \right] dp, \quad (3)$$

where $\mathrm{proj}_{Dz_1(p)}$ *is projection on the span of the vectors $D_1 z_1(p), \ldots, D_{l-1} z_1(p)$ of derivatives of z_1 with respect to $p_1, \ldots, p_{l-1}$.*

PROOF: The result follows by the same reasoning as in the proof of Theorem 1. Changing p by the vector $dp = (dp_1, \ldots, dp_{l-1})$, the point $z_1(p)$ sweeps out an area $Z_1^{(l-1)}(p)\, dp_1 \ldots dp_{l-1}$, where $Z_1^{(l-1)}(p)$ is the $(l-1)$-dimensional volume element at $z_1(p)$, and the volume of the set of weights $a = (a_1, \ldots, a_n)$ which are orthogonal to $x_1(p)$ covered in this movement can be found as this quantity times $|\mathrm{pr}_{Dv(p)}(a)|$, integrated over all $a \in \{\gamma_1(p)\}^\perp$. Using the inequality $Z_1^{(l-1)}(p) \le \|D_1 z_1(p)\| \cdots \|D_{l-1} z_1(p)\|$ we get the expression in (3). □

The bound which can be derived using (3) is rather crude, neglecting the equilibrium conditions in all but one commodity, but it can be used for deriving a bound which exploits the equilibrium property for all commodities. For the evaluation of the expected number of equilibria, the relevant geometric condition is that a is orthogonal to *all* the vectors $z_h(p)$, $h = 1, \ldots, l-1$, where $z_h(p) = (z_{1h}(p), \ldots, z_{l-1h}(p))$. Therefore the righthand side of (3) overstates the expected number of equilibria. We can improve on the bound by reducing integration to the set where the correct geometric condition is satisfied.

Theorem 2. *Let $(\mathcal{C}^0, \mu)$ be a family of economies with l commodities. Then the expected number of equilibria satisfies*

$$\mathsf{E}\,\nu \le \min_{h=1,\dots,l-1} \int_{\triangle_{l-1}} |D_1 z_h(p)| \cdots |D_{l-1} z_h(p)| \left[\int_{\Gamma(p)} |\mathrm{proj}_{D\gamma_h(p)} a| \, d\mu(a) \right] dp,$$

where $\Gamma(p) = \{\gamma_1(p), \dots, \gamma_{l-1}(p)\}^\perp$.

Proof: Apply Lemma 2, which holds for arbitrary $h \in \{1, \dots, l-1\}$, and restrict integration over a to $\Gamma(p)$. □

We may derive a more usable version of the bound using the same method as in the previous section:

Corollary. *Let $(\mathcal{C}^0, \mu)$ be a family of economies with l commodities, with $\mathrm{supp}\,\mu \subset \{a \in \mathbb{R}_+^{\mathcal{C}^0} \mid \sum_{i=1}^n a_i = 1\}$. Then there is a compact subset K of $\mathrm{int}\triangle_{l-1}$ such that $p \notin K$ implies that there is $h \in \{1, \dots, l\}$ with $\zeta_{ih}(p) > 0$, all i, and if*

$$M_K = \min_{h=1,\dots,l-1} \max_{i=1,\dots,l-1} \max_{p \in K} \|D_i z_h(p)\|,$$

then $\mathsf{E}\,\nu \le M^{l-1} m_{l-1}(K)$, where m_{l-1} is $(l-1)$-dimensional Lebesgue measure.

Proof: The existence of a compact set K with the properties stated follows from the monotonicity assumption on the underlying consumers, since for each i and h there is $p_h^i > 0$ such that $\zeta_{ih}(p) > 0$ whenever $p_h \le p_h^i$. Clearly, all equilibrium prices for economies in the family must belong to K. The remaining part of the statement now follows from the theorem. □

While the one-dimensional precise formula for the average number of equilibria can still be put to use in the many-commodity case, it can be seen from the results that this comes at a cost, partly in the form of bounds instead of exact formula, partly as more complicated expressions. Although these drawbacks make applications less simple, it is still possible to extract useful information on particular families of exchange economies from the above results.

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