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Classes with Negative Coefficients and Convex with Respect to other Points

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Abstract

Let $\mathcal{S}$ be the class of functions f which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and a_n a complex number. Let $\mathcal{T}$ denote the class consisting of functions f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ where a_n is a non negative real number. In [6], Sakaguchi introduced the class of analytic functions which are univalent and starlike with respect to symmetric points. Such class is denoted by $S_s^\star$ and satisfies $Re \left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0$ for $z \in D$. Arising from the above introduction, there have been numerous papers on extended and generalised classes of functions. In this paper, we introduce 3 subclasses of $\mathcal{T}$; $C_sT(\alpha, \beta)$, $C_cT(\alpha, \beta)$ and $C_{sc}T(\alpha, \beta)$, consisting of analytic functions with negative coefficients and are respectively convex with respect to symmetric points, convex with respect to conjugate points and convex with respect to symmetric conjugate points. Here, α and β are to satisfy certain constraints. We obtain coefficient conditions for the above classes.

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1 Introduction

Let $\mathcal{S}$ be the class of functions f which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and a_n a complex number. Let $S^\star$ be the subclass of $\mathcal{S}$ consisting of functions starlike in D. It is well known that $f \in S^\star$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in D.$$

Let $S_s^\star$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \quad z \in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in [6]. The class has also been considered in Robertson [5], Stankiewicz [8], Wu [10] and Owa et al. [4]. El-Ashwah and Thomas in [2], introduced two other classes namely the class $S_c^\star$ consisting of functions starlike with respect to conjugate points and $S_{sc}^\star$ consisting of functions starlike with respect to symmetric conjugate points.

In [9], Sudharsan et al. introduced $S_s^\star(\alpha,\beta)$ of functions f analytic and univalent in D given by (1) and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - 1\right| < \beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)} + 1\right|$$

for some $0 \le \alpha \le 1, 0 < \beta \le 1$ and $z \in D$.

However, for this paper, we consider a subclass of $\mathcal{T}$ where $\mathcal{T}$ denotes the class consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

where a_n is a non negative real number.

For $f \in \mathcal{T}$, we define the classes $C_sT(\alpha,\beta), C_cT(\alpha,\beta)$ and $C_{sc}T(\alpha,\beta)$ with α and β satisfying the conditions $0 \le \alpha < 1,\ 0 < \beta < 1$ and $0 \le \frac{2(1-\beta)}{1+\alpha\beta} < 1$.

Definition 1.1 *A function $f \in C_sT(\alpha,\beta)$ is said to be convex with respect to symmetric points if and only if it satisfies*

$$\left|\frac{(zf'(z))'}{(f(z)-f(-z))'} - 1\right| < \beta\left|\frac{\alpha(zf'(z))'}{(f(z)-f(-z))'} + 1\right|$$

for $z \in D$.

Definition 1.2 *A function $f \in C_cT(\alpha,\beta)$ is said to be convex with respect to conjugate points if and only if it satisfies*

$$\left|\frac{(zf'(z))'}{\left(f(z)+\overline{f(\bar{z})}\right)'}-1\right|<\beta\left|\frac{\alpha(zf'(z))'}{\left(f(z)+\overline{f(\bar{z})}\right)'}+1\right|$$

for $z \in D$.

Definition 1.3 *A function $f \in C_{sc}T(\alpha,\beta)$ is said to be convex with respect to symmetric conjugate points if and only if it satisfies*

$$\left|\frac{(zf'(z))'}{\left(f(z)-\overline{f(-\bar{z})}\right)'}-1\right|<\beta\left|\frac{\alpha(zf'(z))'}{\left(f(z)-\overline{f(-\bar{z})}\right)'}+1\right|$$

for $z \in D$.

At this point, we would like to note that the above conditions imposed on α and β, are necessary to ensure these classes form subclasses of $\mathcal{S}$.

2 Preliminaries

We first state preliminary results, required for proving our main results.

Lemma 2.1 *If $f \in \mathcal{T}$ then $\sum_{n=2}^{\infty} n(n\alpha+(1-(-1)^n))\, a_n\, |z|^{n-1} < 2+\alpha$.*

Proof. Since $f \in \mathcal{T}$, $\sum_{n=2}^{\infty} n\, |a_n|\, |z|^{n-1} < 1$ and $\sum_{n=2}^{\infty} n^2\, |a_n|\, |z|^{n-1} < 1$(see Silverman [7]). Thus we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n\alpha+(1-(-1)^n))\, a_n\, |z|^{n-1} &= \sum_{n=2}^{\infty} n^2\alpha\, a_n\, |z|^{n-1} + \sum_{n=2}^{\infty} n(1-(-1)^n)\, a_n\, |z|^{n-1} \\
&= \alpha\sum_{n=2}^{\infty} n^2\, a_n\, |z|^{n-1} + 2\sum_{n=1}^{\infty} na_{2n+1}\, |z|^{2n} \\
&\le \alpha\sum_{n=2}^{\infty} n^2\, a_n\, |z|^{n-1} + 2\sum_{n=1}^{\infty} (2n+1)a_{2n+1}\, |z|^{2n} \\
&\le \alpha\sum_{n=2}^{\infty} n^2\, a_n\, |z|^{n-1} + 2\sum_{n=2}^{\infty} na_n\, |z|^{n-1} \\
&< \alpha+2.
\end{aligned}$$

Lemma 2.2 *If $f \in \mathcal{T}$ then $\sum_{n=2}^{\infty} n(n\alpha+2)\, a_n\, |z|^{n-1} < 2+\alpha$.*

Proof. Since $f \in \mathcal{T}$ then

$$\begin{aligned}\sum_{n=2}^{\infty} n(n\alpha+2)\ a_n\ |z|^{n-1} &= \sum_{n=2}^{\infty} n^2\alpha\ a_n\ |z|^{n-1} + \sum_{n=2}^{\infty} 2n\ a_n\ |z|^{n-1} \\ &= \alpha\sum_{n=2}^{\infty} n^2\ a_n\ |z|^{n-1} + 2\sum_{n=2}^{\infty}\ na_n\ |z|^{n-1} \\ &< \alpha+2.\end{aligned}$$

3 Results

In this section, we give results concerning the coefficients estimates of the 3 main classes.

Theorem 3.1 *$f \in C_sT(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n^2}{\beta(2+\alpha)-1} + \frac{n\beta(1-(-1)^n)-n(1-(-1)^n)}{\beta(2+\alpha)-1}\right) a_n \leq 1. \tag{3}$$

Proof. We adopt the method used by Clunie and Keogh [1] and Owa [3]. First we prove the 'if' part. We begin by considering
$|(zf'(z))' - (f(z)-f(-z))'| - \beta|\alpha(zf'(z))' + (f(z)-f(-z))'|$

$$\begin{aligned} &= \left|-1-\sum_{n=2}^{\infty} n(n-(1-(-1)^n))a_n z^{n-1}\right| - \beta\left|(2+\alpha)-\sum_{n=2}^{\infty} n(n\alpha+(1-(-1)^n))a_n z^{n-1}\right| \\ &\leq \sum_{n=2}^{\infty} n(n-(1-(-1)^n))a_n r^{n-1} + 1 - \beta(2+\alpha) + \sum_{n=2}^{\infty} n\beta(n\alpha+(1-(-1)^n))a_n r^{n-1} \\ &< \sum_{n=2}^{\infty} n(n-(1-(-1)^n))a_n + 1 - \beta(2+\alpha) + \sum_{n=2}^{\infty} n\beta(n\alpha+(1-(-1)^n))a_n \\ &= \sum_{n=2}^{\infty}((1+\beta\alpha)n^2 + n\beta(1-(-1)^n) - n(1-(-1)^n))a_n - (\beta(2+\alpha)-1) \\ &\leq 0 \text{ by } (3).\end{aligned}$$

Thus,

$$\left|\frac{\frac{(zf'(z))'}{(f(z)-f(-z))'}-1}{\frac{\alpha(zf'(z))'}{(f(z)-f(-z))'}+1}\right| < \beta$$

and hence $f \in C_sT(\alpha,\beta)$. To prove the 'only if' part, we write

$$\left|\frac{\frac{(zf'(z))'}{(f(z)-f(-z))'}-1}{\frac{\alpha(zf'(z))'}{(f(z)-f(-z))'}+1}\right| = \left|\frac{-1-\sum_{n=2}^{\infty} n(n-(1-(-1)^n))a_n z^{n-1}}{2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+(1-(-1)^n))a_n z^{n-1}}\right| < \beta.$$

We note that since f is analytic, continuous and non constant in $\mathcal{D}$, the maximum modulus principle gives

$$\left|\frac{-1-\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n z^{n-1}}{2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n z^{n-1}}\right|$$

$$\begin{aligned} &= \frac{|1+\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n z^{n-1}|}{|2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n z^{n-1}|} \\ &\leq \frac{1+\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n |z|^{n-1}}{2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n |z|^{n-1}} \\ &\leq \frac{1+\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n r^{n-1}}{2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n r^{n-1}}\ , \end{aligned}$$

where $\sum_{n=2}^{\infty} n(n\alpha+(1-(-1)^n))\ a_n\ |z|^{n-1} < 2+\alpha$ from Lemma 2.1. Since $f \in C_sT(\alpha,\beta)$ and $0<r<1$, we obtain

$$\left\{\frac{1+\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n r^{n-1}}{2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n r^{n-1}}\right\} < \beta. \tag{4}$$

Now letting $r \to 1$ in (4), and using Lemma 2.1,

$$1+\sum_{n=2}^{\infty}\ n(n-(1-(-1)^n))a_n \leq \beta\left(2+\alpha-\sum_{n=2}^{\infty}\ n(n\alpha+(1-(-1)^n))a_n\right)$$

and hence

$$\sum_{n=2}^{\infty}\left\{\frac{(1+\beta\alpha)n^2}{\beta(2+\alpha)-1}+\frac{n\beta(1-(-1)^n)-n(1-(-1)^n)}{\beta(2+\alpha)-1}\right\}a_n \leq 1$$

as required. This completes the proof of the theorem.

The result in Theorem 3.1 is sharp for functions given by

$$f_n(z) = z - \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2+n\beta(1-(-1)^n)-n(1-(-1)^n)}\ z^n, \quad n \geq 2.$$

Corollary 3.1 *If $f \in C_sT(\alpha,\beta)$ then*

$$a_n \leq \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2+n\beta(1-(-1)^n)-n(1-(-1)^n)}\ , \quad n \geq 2.$$

For $C_cT(\alpha,\beta)$, we state the result in Theorem 3.2. Similar method is used here as that in Theorem 3.1.

Theorem 3.2 *$f \in C_cT(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n^2}{\beta(2+\alpha)-1}+\frac{2n(\beta-1)}{\beta(2+\alpha)-1}\right)a_n \leq 1. \tag{5}$$

Proof. As before, we consider
$|(zf'(z))' - \left(f(z)+\overline{f(\bar{z})}\right)'| - \beta|\alpha(zf'(z))' + \left(f(z)+\overline{f(\bar{z})}\right)'|$

$$\begin{aligned}
&= \left|-1-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}+\sum_{n=2}^{\infty} n\overline{a_n}z^{n-1}\right| - \beta\left|(2+\alpha)-\sum_{n=2}^{\infty} n(n\alpha+1)a_n z^{n-1}-\sum_{n=2}^{\infty} n\overline{a_n}z^{n-1}\right| \\
&= \left|-1-\sum_{n=2}^{\infty} n(n-2)a_n z^{n-1}\right| - \beta\left|(2+\alpha)-\sum_{n=2}^{\infty} n(n\alpha+2)a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty} n(n-2)a_n r^{n-1}+1-\beta(2+\alpha)+\sum_{n=2}^{\infty} n\beta(n\alpha+2)a_n r^{n-1} \\
&< \sum_{n=2}^{\infty} n(n-2)a_n+1-\beta(2+\alpha)+\sum_{n=2}^{\infty} n\beta(n\alpha+2)a_n \\
&= \sum_{n=2}^{\infty}((1+\beta\alpha)n^2+2n(\beta-1))a_n-(\beta(2+\alpha)-1) \\
&\leq 0 \text{ by } (5).
\end{aligned}$$

Hence $f \in C_cT(\alpha,\beta)$. For the 'only if' part, the maximum modulus principle gives

$$\begin{aligned}
\left|\frac{-1-\sum_{n=2}^{\infty} n(n-2)a_n z^{n-1}}{2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+2)a_n z^{n-1}}\right| &\leq \frac{1+\sum_{n=2}^{\infty} n(n-2)a_n|z|^{n-1}}{2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+2)a_n|z|^{n-1}} \\
&\leq \frac{1+\sum_{n=2}^{\infty} n(n-2)a_n r^{n-1}}{2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+2)a_n r^{n-1}},
\end{aligned}$$

where $\sum_{n=2}^{\infty} n(n\alpha+2)\, a_n\, |z|^{n-1} < 2+\alpha$ from Lemma 2.2. Since $f \in C_cT(\alpha,\beta)$ and $0<r<1$, we obtain

$$\left\{\frac{1+\sum_{n=2}^{\infty} n(n-2)a_n r^{n-1}}{2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+2)a_n r^{n-1}}\right\} < \beta. \tag{6}$$

Now letting $r \to 1$ in (6), and using Lemma 2.2,

$$1+\sum_{n=2}^{\infty} n(n-2)a_n \leq \beta\left(2+\alpha-\sum_{n=2}^{\infty} n(n\alpha+2)a_n\right)$$

and hence

$$\sum_{n=2}^{\infty}\left\{\frac{(1+\beta\alpha)n^2}{\beta(2+\alpha)-1}+\frac{2n(\beta-1)}{\beta(2+\alpha)-1}\right\}a_n \leq 1.$$

This completes the proof of the theorem.

The result in Theorem 3.2 is sharp for function f_n given by

$$f_n(z) = z - \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2 + 2n(\beta-1)}\, z^n, \quad n \geq 2.$$

Corollary 3.2 *If* $f \in C_cT(\alpha,\beta)$ *then*

$$a_n \leq \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2 + 2n(\beta-1)}\,, \quad n \geq 2.$$

Finally, we give similar results for functions which belong to $C_{sc}T(\alpha,\beta)$. Method of proving Theorem 3.3 is similar as that of Theorem 3.1 and 3.2.

Theorem 3.3 $f \in C_{sc}T(\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n^2}{\beta(2+\alpha)-1} + \frac{n\beta(1-(-1)^n) - n(1-(-1)^n)}{\beta(2+\alpha)-1}\right) a_n \leq 1.$$

The result in Theorem 3.3 is sharp for functions given by

$$f_n(z) = z - \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2 + n\beta(1-(-1)^n) - n(1-(-1)^n)}\, z^n, \quad n \geq 2.$$

Corollary 3.3 *If* $f \in C_{sc}T(\alpha,\beta)$ *then*

$$a_n \leq \frac{\beta(2+\alpha)-1}{(1+\beta\alpha)n^2 + n\beta(1-(-1)^n) - n(1-(-1)^n)}\,, \quad n \geq 2.$$

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