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Harmonic Functions which are Starlike of Order β with Respect to other Points

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Abstract

Let $\mathcal{H}$ denote the class of functions f which are harmonic and univalent in the open unit disc $D = \{z : |z| < 1\}$. This paper defines and investigates a family of complex-valued harmonic functions that are orientation preserving and univalent in $\mathcal{D}$ and are related to the functions starlike of order $\beta(0 \leq \beta < 1)$, with respect to other points. We obtain growth result, extreme points, convolution and convex combinations for the above harmonic functions.

Keywords: harmonic functions, starlike of order β with respect to conjugate points, coefficient estimates

1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain E is said to be harmonic in E if both u and v are real harmonic in E. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re\ (U)$ and $v = Im\ (V)$. Then

$$f(z) = h(z) + \overline{g(z)}$$

where h and g are, respectively, the analytic functions $(U+V)/2$ and $(U-V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by

$$J_f = |h'(z)|^2 - |g'(z)|^2.$$

The mapping $z \mapsto f(z)$ is orientation preserving and locally one-to-one in E if and only if $J_f > 0$ in E. See also Clunie and Sheil-Small [1]. The function $f = h + \overline{g}$ is said to be harmonic univalent in E if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one-to-one in E. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let $\mathcal{H}$ denote the class of functions $f = h + \overline{g}$ which are harmonic and univalent in $\mathcal{D}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

Also let $\overline{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{2}$$

In [2], Darus et al. define a new class of functions as follows:

Definition 1.1 *Let $f \in \mathcal{H}$. Then $f \in \mathcal{HS}_s^\star(\beta)$ is said to be harmonic starlike of order β, with respect to symmetric points, if and only if, for $0 \le \beta < 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \le r < 1$ and $0 \le \theta < 2\pi$,*

$$Re\ \left\{\frac{2zf'(z)}{z'(f(z) - f(-z))}\right\} \ge \beta\ ,$$

and $\overline{\mathcal{H}}\mathcal{S}_s^\star(\beta) = \mathcal{HS}_s^\star(\beta) \cap \overline{H}$.

Then, in [3], Halim et al. define new classes of functions as follows:

Definition 1.2 *Let $f \in \mathcal{H}$. Then for $0 \le \beta < 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \le r < 1$ and $0 \le \theta < 2\pi$,*

(i) *$f \in \mathcal{HS}^\star_c(\beta)$ is said to be harmonic starlike of order β, w.r.t conjugate points, if and only if,*

$$Re\left\{\frac{2zf'(z)}{z'(f(z)+\overline{f(\overline{z})})}\right\} \ge \beta; \tag{3}$$

(ii) *$f \in \mathcal{HS}^\star_{sc}(\beta)$ is said to be harmonic starlike of order β, w.r.t symmetric conjugate points, if and only if,*

$$Re\left\{\frac{2zf'(z)}{z'(f(z)-\overline{f(-\overline{z})})}\right\} \ge \beta.$$

Also, we let $\overline{\mathcal{H}}\mathcal{S}^\star_c(\beta) = \mathcal{HS}^\star_c(\beta) \cap \overline{H}$ and $\overline{\mathcal{H}}\mathcal{S}^\star_{sc}(\beta) = \mathcal{HS}^\star_{sc}(\beta) \cap \overline{H}$.

The following theorem proved by Halim et al. in [3] will be used throughout in this paper.

Theorem 1.1 *([3]) Let $f = h + \overline{g}$ with h and g of the form (1). If*

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}|b_n| \le 1, \quad 0 \le \beta < 1, \tag{4}$$

then f is harmonic, sense-preserving, univalent in $\mathcal{D}$ and $f \in \mathcal{HS}^\star_c(\beta)$. Condition (4) is also necessary if $f \in \overline{\mathcal{H}}\mathcal{S}^\star_c(\beta) = \mathcal{HS}^\star_c(\beta) \cap \overline{H}$.

2 Results

We begin the results with a growth result for functions in $\overline{\mathcal{H}}\mathcal{S}^\star_c(\beta)$.

Theorem 2.1 *If $f \in \overline{\mathcal{H}}\mathcal{S}^\star_c(\beta)$ then*

$$|f(z)| \le (1+|b_1|)r + \left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2, \quad |z| = r < 1$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2, \quad |z| = r < 1.$$

Proof. Let $f \in \overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^2 \\
&= (1+|b_1|)r + \frac{1-\beta}{2-\beta}\sum_{n=2}^{\infty}\left(\frac{2-\beta}{1-\beta}|a_n| + \frac{2-\beta}{1-\beta}|b_n|\right)\ r^2 \\
&\leq (1+|b_1|)r + \\
&\quad \frac{1-\beta}{2-\beta}\sum_{n=2}^{\infty}\left(\frac{n-\beta}{1-\beta}|a_n| + \frac{n+\beta}{1-\beta}|b_n|\right)\ r^2 \\
&\leq (1+|b_1|)r + \frac{1-\beta}{2-\beta}\left(1 - \frac{1+\beta}{1-\beta}|b_1|\right)\ r^2 \\
&= (1+|b_1|)r + \left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)\ r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^2 \\
&= (1-|b_1|)r - \frac{1-\beta}{2-\beta}\sum_{n=2}^{\infty}\left(\frac{2-\beta}{1-\beta}|a_n| + \frac{2-\beta}{1-\beta}|b_n|\right)\ r^2 \\
&\geq (1-|b_1|)r - \\
&\quad \frac{1-\beta}{2-\beta}\sum_{n=2}^{\infty}\left(\frac{n-\beta}{1-\beta}|a_n| + \frac{n+\beta}{1-\beta}|b_n|\right)\ r^2 \\
&\geq (1-|b_1|)r - \frac{1-\beta}{2-\beta}\left(1 - \frac{1+\beta}{1-\beta}|b_1|\right)\ r^2 \\
&= (1-|b_1|)r - \left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)\ r^2. \quad \square
\end{aligned}$$

The bounds given in Theorem 2.1 for the functions $f = h + \overline{g}$ of the form (2) also hold for functions of the form (1) if the coefficient condition (4) is satisfied. The upper bound given for $f \in \overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1|\overline{z} + \left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)\overline{z}^{\,2}, \quad |b_1| \leq \frac{1-\beta}{1+\beta}.$$

Next, we determine the extreme points of closed hulls of $\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$ denoted by $clco\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$.

Theorem 2.2 *$f \in clco\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$ if and only if $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ where*

$$h_1(z) = z,\ h_n(z) = z - \frac{1-\beta}{n-\beta}\, z^n\ (n = 2, 3, ...),$$

$$g_n(z) = z + \frac{1-\beta}{n+\beta}\, \bar{z}^{\,n}\ (n = 1, 2, ...),$$

$\sum_{n=1}^{\infty}(X_n + Y_n) = 1,\ X_n \geq 0$ and $Y_n \geq 0$.

Proof. For h_n and g_n as given above, we may write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \\ &= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{1-\beta}{n-\beta}X_n z^n + \sum_{n=1}^{\infty}\frac{1-\beta}{n+\beta}Y_n \bar{z}^{\,n} \\ &= z - \sum_{n=2}^{\infty}\frac{1-\beta}{n-\beta}X_n z^n + \sum_{n=1}^{\infty}\frac{1-\beta}{n+\beta}Y_n \bar{z}^{\,n}. \end{aligned}$$

Then

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}|b_n|$$

$$\begin{aligned} &= \sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}\left(\frac{1-\beta}{n-\beta}X_n\right) \\ &\quad + \sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}\left(\frac{1-\beta}{n+\beta}Y_n\right) \\ &= \sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n \\ &= 1 - X_1 \\ &\leq 1. \end{aligned}$$

Therefore $f \in clco\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$.

Conversely, suppose that $f \in clco\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$. Set

$$X_n = \frac{n-\beta}{1-\beta}|a_n|\ , (n = 2, 3, 4, ...),$$

and

$$Y_n = \frac{n+\beta}{1-\beta}|b_n|\ , (n = 1, 2, 3, ...),$$

where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned} f(z) &= h(z)+\overline{g(z)} \\ &= z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n} \\ &= z-\sum_{n=2}^{\infty}\frac{1-\beta}{n-\beta}X_nz^n+\sum_{n=1}^{\infty}\frac{1-\beta}{n+\beta}Y_n\bar{z}^{\,n} \\ &= z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n \\ &= \sum_{n=1}^{\infty}(X_nh_n+Y_ng_n). \quad \square \end{aligned}$$

For harmonic functions $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z)=z-\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=1}^{\infty}|B_n|\bar{z}^{\,n}$, we define the convolution of f and F as

$$(f\star F)(z)=z-\sum_{n=2}^{\infty}|a_nA_n|z^n+\sum_{n=1}^{\infty}|b_nB_n|\bar{z}^{\,n}. \tag{5}$$

In the next theorem, we examine the convolution properties of the class $\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$.

Theorem 2.3 *For $0\le\alpha\le\beta<1$, let $f\in\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$ and $F\in\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\alpha)$. Then $(f\star F)\in\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)\subset\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\alpha)$.*

Proof. Write $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z)=z-\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=1}^{\infty}|B_n|\bar{z}^{\,n}$. Then the convolution of f and F is given by (5).

Note that $|A_n|\le 1$ and $|B_n|\le 1$ since $F\in\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\alpha)$. Then we have

$$\sum_{n=2}^{\infty}(n-\beta)|a_n||A_n|+\sum_{n=1}^{\infty}(n+\beta)|b_n||B_n|$$

$$\le\sum_{n=2}^{\infty}(n-\beta)|a_n|+\sum_{n=1}^{\infty}(n+\beta)|b_n|.$$

Therefore $(f\star F)\in\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)\subset\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\alpha)$ since the right hand side of the above inequality is bounded by $1-\beta$ while $1-\beta\le 1-\alpha$. $\square$

Now, we determine the convex combination properties of the members of $\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$.

Theorem 2.4 *The class $\overline{\mathcal{H}}\mathcal{S}_c^{\star}(\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{c}(\beta)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^{\,n}.$$

For $\sum_{i=1}^{\infty} c_i = 1,\ 0 \leq c_i \leq 1$, the convex combinations of f_i may be written as

$$\begin{aligned}
\sum_{i=1}^{\infty} c_i f_i(z) &= c_1 z - \sum_{n=2}^{\infty} c_1 |a_{n,1}| z^n + \sum_{n=1}^{\infty} c_1 |b_{n,1}| \bar{z}^{\,n} + c_2 z - \sum_{n=2}^{\infty} c_2 |a_{n,2}| z^n + \sum_{n=1}^{\infty} c_2 |b_{n,2}| \bar{z}^{\,n} ... \\
&= z \sum_{i=1}^{\infty} c_i - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\,n} \\
&= z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\,n}.
\end{aligned}$$

Next, consider

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left((n-\beta) \left| \sum_{i=1}^{\infty} c_i |a_{n,i}| \right| \right) + \sum_{n=1}^{\infty} \left((n+\beta) \left| \sum_{i=1}^{\infty} c_i |b_{n,i}| \right| \right) \\
&= c_1 \sum_{n=2}^{\infty} (n-\beta)|a_{n,1}| + ... + c_m \sum_{n=2}^{\infty} (n-\beta)|a_{n,m}| + ... \\
&\quad + c_1 \sum_{n=1}^{\infty} (n+\beta)|b_{n,1}| + ... + c_m \sum_{n=1}^{\infty} (n+\beta)|b_{n,m}| + ... \\
&= \sum_{i=1}^{\infty} c_i \left\{ \sum_{n=2}^{\infty} (n-\beta)|a_{n,i}| + \sum_{n=1}^{\infty} (n+\beta)|b_{n,i}| \right\}.
\end{aligned}$$

Now, $f_i \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{c}(\beta)$, therefore from Theorem 1.1, we have

$$\sum_{n=2}^{\infty} (n-\beta)|a_{n,i}| + \sum_{n=1}^{\infty} (n+\beta)|b_{n,i}| \leq 2(1-\beta)\ .$$

Hence

$\sum_{n=2}^{\infty} \left((n-\beta) \left| \sum_{i=1}^{\infty} c_i |a_{n,i}| \right| \right) + \sum_{n=1}^{\infty} \left((n+\beta) \left| \sum_{i=1}^{\infty} c_i |b_{n,i}| \right| \right)$

$\leq (1-\beta) \sum_{i=1}^{\infty} c_i$

$= 1-\beta.$

By using Theorem 1.1 again, we have $\sum_{i=1}^{\infty} c_i f_i \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{c}(\beta)$. □

For $\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta) = \mathcal{H}\mathcal{S}^{\star}_{sc}(\beta) \cap \overline{H}$, we state the results as follows. Method of proving are left out since similar method is used here.

Theorem 2.5 *Let $f = h + \overline{g}$ be of the form (1). If*

$$\sum_{n=2}^{\infty} \frac{(2n - \beta(1-(-1)^n))}{2(1-\beta)}|a_n| + \sum_{n=1}^{\infty} \frac{(2n + \beta(1-(-1)^n))}{2(1-\beta)}|b_n| \leq 1,$$

then f is harmonic, sense-preserving, univalent in $\mathcal{D}$ and $f \in \mathcal{HS}^{\star}_{sc}(\beta)$.

Theorem 2.6 *Let $f = h + \overline{g}$ with h and g of the form (2). Then $f \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(2n - \beta(1-(-1)^n))}{2(1-\beta)}|a_n| + \sum_{n=1}^{\infty} \frac{(2n + \beta(1-(-1)^n))}{2(1-\beta)}|b_n| \leq 1.$$

Theorem 2.7 *If $f \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ then*

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\beta}{2} - \frac{1+\beta}{2}|b_1|\right) r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\beta}{2} - \frac{1+\beta}{2}|b_1|\right) r^2, \quad |z| = r < 1.$$

Next, we determine the extreme points of closed hulls of $\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ denoted by $clco\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$.

Theorem 2.8 *$f \in clco\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ if and only if $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ where*

$$h_1(z) = z, \; h_n(z) = z - \frac{2(1-\beta)}{2n - \beta(1-(-1)^n)} \; z^n \; (n = 2, 3, ...),$$

$$g_n(z) = z + \frac{2(1-\beta)}{2n + \beta(1-(-1)^n)} \; \bar{z}^{\,n} \; (n = 1, 2, ...),$$

$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \; X_n \geq 0$ and $Y_n \geq 0$.

In the next theorem, we examine the convolution properties of the class $\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$.

Theorem 2.9 *For $0 \leq \alpha \leq \beta < 1$, let $f \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ and $F \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\alpha)$. Then $f \star F \in \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta) \subset \overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\alpha)$.*

Now, we determine the convex combination properties of the members of $\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$.

Theorem 2.10 *The class $\overline{\mathcal{H}}\mathcal{S}^{\star}_{sc}(\beta)$ is closed under convex combination.*

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