Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 34, 1665 - 1677

Fuzzy Almost Continuous Functions

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Abstract

In this paper, we further study properties of fuzzy almost continuous functions in Singal's and Hussain's sense and establish a condition for their equivalence which is an improvement of Theorem 5.5 of [2]. Some divergences from straightforward fuzzification of General Topology have also been noted in Example 1 and Theorem 4. We also define fuzzy almost weakly continuous and fuzzy nearly almost continuous functions and study their interesting properties and characterizations.

Mathematics Subject Classification: Primary 54A40

Keywords: Fuzzy almost continuous; Singal sense; Hussain sense; Fuzzy weakly continuous; Fuzzy nearly almost continuous

1 Introduction

In 1965, L. Zadeh [21] introduced the concept of fuzzy sets. Chang, Kerre, Lowen, Katsaras, Wong, Warren and many others contributed a lot to the field of Fuzzy Topology. In recent years Fuzzy Topology has been found to be very useful in solving many practical problems. Shihong Du *et. al.* [5] are currently working to fuzzify the 9-intersection Egenhofer model ([6] [7]) for describing topological relations in Geographic Information Systems (GIS) query. In ([13] [14]), El-Naschie has shown that the notion of Fuzzy Topology is applicable to quantum particle physics and quantum gravity in connection with String Theory and e^{∞} Theory. Tang [18] has used a slightly changed version of Chang's fuzzy topological space to model spatial objects for GIS databases and Structured Query Language (SQL) for GIS.

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In 1981, Azad [4] gave fuzzy version of the concepts given by Levine [11] and thus initiated the study of weak forms of several notions in fuzzy topological spaces. In this paper, first we define fuzzy regularly open (closed) set which is a weaker notion of fuzzy regularly open set defined by K. Azad [4] and prove that the notions of fuzzy regularly open set and fuzzy semi-preopen set, defined and studied by Zhong [19] are equivalent. It is also shown that in the class of injective functions, fuzzy almost open (closed) in Nanda's sense and fuzzy almost quasi-compact functions are equivalent. In terms of graph and projections, some intresting characterizations and properties of fuzzy almost continuous functions in Singal's sense are given. Moreover fuzzy almost continuous in Hussain's sense, fuzzy almost weakly continuous, fuzzy nearly almost open (closed) functions have been defined and their several characterizations and properties have been obtained. Finally, their equivalences have been established under certain conditions.

2 Preliminaries

Definition 1 *[4] A fuzzy set λ is said to be fuzzy regularly open (resp. fuzzy regularly closed) (briefly f.r.o (resp. f.r.c)) if $IntCl\lambda = \lambda$ (resp. $ClInt\lambda = \lambda$).*

Definition 2 *[4] A mapping $f : X \to Y$ from an fts X to another fts Y is called a fuzzy almost continuous mapping in Singal's sense (briefly, f.a.c.S), if $f^{-1}(\lambda)$ is fuzzy open in X, for each fuzzy regularly open set λ in Y.*

Remark 1 *Every fuzzy continuous function $f : X \to Y$ at a point $x \in X$, is f.a.c.S. at x, the converse is not true as shown in [4] (Example 7.5).*

Definition 3 *[4] A mapping $f : X \to Y$ is said to be fuzzy open if $f(\lambda)$ is fuzzy open set in Y for every fuzzy open set λ in X.*

Theorem 1 *[20] Let $f : X \to Y$ be a fuzzy open function. Then for every fuzzy set β in Y, $f^{-1}(Cl\beta) \leq Clf^{-1}(\beta)$.*

Theorem 2 *[2] Let $f : X \to Y$ be a f.a.c.S. and η a fuzzy open set in Y. If $x_\alpha \in Clf^{-1}(\eta)$, then $f(x_\alpha) \in Cl\eta$.*

Definition 4 *A fuzzy set λ in an fts X is said to be*
(1) *fuzzy α-open [16], if $\lambda \leq IntClInt\lambda$ (resp. α-closed, if $\lambda \geq ClIntCl\lambda$),*
(2) *fuzzy preopen [16], if $\lambda \leq IntCl\lambda$ (resp. preclosed, if $ClInt\lambda \leq \lambda$),*
(3) *fuzzy semi-open (resp. fuzzy semi-closed) [4], if there exists a fuzzy open (resp. fuzzy closed)set μ such that $\mu \leq \lambda \leq Cl\mu$ (resp. $Int\mu \leq \lambda \leq \mu$).*

It is known [4] that λ is fuzzy semi-open (resp. fuzzy semi-closed) if and only if $\lambda \leq ClInt\lambda$ (resp. $IntCl\lambda \leq \lambda$).

Remark 2 (1) *Every fuzzy α-open (resp. fuzzy α-closed) set is a fuzzy semi-open (resp. fuzzy semi-closed) set.*
(2) *Every fuzzy α-open (resp. fuzzy α-closed) set is a fuzzy preopen (resp. fuzzy pre-closed) set.*

The classes of all fuzzy α-open, fuzzy preopen, fuzzy semi-open and fuzzy regularly open sets of an fts X are denoted as $\alpha(X)$, $FPO(X)$, $FSO(X)$ and $FRO(X)$ respectively.

Definition 5 *[17] Let λ be a fuzzy set in an fts X. Then its p-closure and p-interior are denoted and defined as:*

$$\begin{aligned} pCl\lambda &= \wedge\{\mu|\mu \geq \lambda,\ \mu \text{ is a fuzzy pre-closed set of } X\}, \\ pInt\lambda &= \vee\{\nu|\nu \leq \lambda,\ \nu \text{ is a fuzzy pre-open set of } X\}. \end{aligned}$$

The definitions of sCl, $sInt$, αCl and αInt are similar.

Theorem 3 *For any fuzzy set λ in an fts X, the following hold:*
(1) $sCl\lambda = \lambda \vee IntCl\lambda$.*[10]*
(2) $sInt\lambda = \lambda \wedge ClInt\lambda$.*[10]*
(3) $\alpha Cl\lambda \geq \lambda \vee ClIntCl\lambda$.
(4) $\alpha Int\lambda \leq \lambda \wedge IntClInt\lambda$.
(5) $pCl\lambda \geq \lambda \vee ClInt\lambda$.*[9]*
(6) $pInt\lambda \leq \lambda \wedge IntCl\lambda$.*[9]*

Proof. (3) Since $\alpha Cl\lambda$ is fuzzy α-closed, we have $ClIntCl\alpha Cl\lambda \leq \alpha Cl\lambda$. Therefore $ClIntCl\lambda \leq \alpha Cl\lambda$, and hence $\lambda \vee ClIntCl\lambda \leq \alpha Cl\lambda$. ■

3 Fuzzy Almost Continuous Functions in Singal's Sense

Definition 6 *[19] A fuzzy set λ in an fts X is said to be fuzzy semi-preopen, if there exists a fuzzy preopen set μ such that $\mu \leq \lambda \leq Cl\mu$. The class of all the fuzzy semi-preopen sets is denoted as $FSPO(X)$.*

In General Topology, we have:

Lemma 1 *[3]*
(1) *For a subset A and an open set B in a space X, $ClA \cap B \subseteq Cl(A \cap B)$.*
(2) *For a subset A and a closed set F in a space X, $Int(A \cup F) \subseteq IntA \cup F$.*

Andrijvic used this lemma to prove Theorem 2.4 [3]. Interestingly this lemma does not hold true in Fuzzy Topology as the following example shows:

Example 1 *Let $X = \{a, b\}$ be a set and*
$\tau = \{\Phi, \{a_{.3}, b_{.7}\}, \{a_{.5}, b_{.5}\}, \{a_{.3}, b_{.5}\}, \{a_{.5}, b_{.7}\}, 1_X\}$, *the fuzzy topology on X. Choose $\lambda = \{a_{.9}, b_{.3}\}$ and a fuzzy open set $\mu = \{a_{.3}, b_{.5}\}$. Then calculations show that*

$$\begin{aligned} Cl\lambda \wedge \mu &= \{a_{.3}, b_{.5}\} \nleq \{a_{.5}, b_{.3}\} = Cl(\lambda \wedge \mu). \\ Int(\lambda \vee \mu) &= \{a_{.5}, b_{.5}\} \nleq \{a_{.3}, b_{.5}\} = Int\lambda \vee \mu. \end{aligned}$$

Consequently, the exact analogue of Theorem 2.4 of [3] is not true in fuzzy settings. Instead we have:

Theorem 4 *For any fuzzy set λ in an fts X, if λ is fuzzy semi-preopen, then $Cl\lambda$ is fuzzy regularly closed.*

Proof. By Theorem 2.3 [15], if λ is fuzzy semi-preopen, then $\lambda \leq ClIntCl\lambda$. This gives $Cl\lambda \leq ClIntCl\lambda$. But $ClIntCl\lambda \leq Cl\lambda$. Therefore, we have $ClIntCl\lambda = Cl\lambda$. This implies $Cl\lambda \in FRC(X)$. ■

Theorem 5 *[4] Let $f : X \to Y$ be a function. Then the following are equivalent:*
(1) *f is f.a.c.S.*
(2) *$f^{-1}(\lambda)$ is a fuzzy closed set in X, for every fuzzy regularly closed set λ of Y.*

Using Theorems 5 and 4, we give a characerization of f.a.c.S. functions:

Theorem 6 *The following are equivalent for a function $f : X \to Y$:*
(1) *f is f.a.c.S.*
(2) *$Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every $\nu \in FSPO(Y)$.*
(3) *$Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.*
(4) *$f^{-1}(\nu) \leq Intf^{-1}(IntCl\nu)$, for every $\nu \in FPO(Y)$.*

Proof. (1) ⇒ (2) Let $\nu \in FSPO(Y)$. By Theorem 4, $Cl\nu$ is fuzzy regularly closed in Y. Since f is f.a.c.S., then by Theorem 5, $f^{-1}(Cl\nu)$ is fuzzy closed in X and we obtain $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.

(2) ⇒ (3) This is obvious, since each fuzzy semi-open set is fuzzy semi-preopen set.

(3) ⇒ (1) Let $\nu \in FRC(Y)$. Then $\nu = ClInt\nu$ and hence $\nu \in FSO(Y)$. Therefore, we have $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu) = f^{-1}(\nu)$. Hence $f^{-1}(\nu)$ is fuzzy closed and by Theorem 5 f is f.a.c.S.

(1) ⇒ (4) Let $\nu \in FPO(Y)$. Then $\nu \leq IntCl\nu$ and $IntCl\nu$ is fuzzy regularly open. Since f is f.a.c.S. by Theorem 5, $f^{-1}(Int\ Cl\nu)$ is fuzzy open in X and hence $f^{-1}(\nu) \leq f^{-1}(IntCl\nu) = Intf^{-1}(IntCl\nu)$.

(4) $\Rightarrow$ (1) Let ν be a fuzzy regularly open set in Y. Then $\nu \in FPO(Y)$ and hence $f^{-1}(\nu) \leq Intf^{-1}(IntCl\nu) = Intf^{-1}(\nu)$. Therefore, $f^{-1}(\nu)$ is fuzzy open in X and f is f.a.c.S. ■

The following is an immediate consequence of Definition 5.

Lemma 2 *Let x_α be a fuzzy point in an fts X. Then, $x_\alpha \in pCl\lambda$ if and only if $\lambda \wedge \nu \neq \Phi$, for every fuzzy preopen set ν in X such that $x_\alpha \in \nu$.*

We use Lemma 2 and prove:

Theorem 7 *If λ is a fuzzy semi-open set in an fts X, then $pCl\lambda = Cl\lambda$.*

Proof. Clearly, $pCl\psi \leq Cl\psi$, for every fuzzy set ψ in X. To prove $Cl\lambda \leq pCl\lambda$ for a fuzzy semi-open set λ, let $x_\alpha \in Cl\lambda$ and $x_\alpha \in \nu$, where ν is a fuzzy preopen set in X. Then $x \in \nu \leq IntCl\nu$ and hence $\lambda \wedge IntCl\nu \neq \Phi$. Since λ is fuzzy semi-open, $\lambda \wedge IntCl\nu \leq ClInt\lambda \wedge IntCl\nu \leq Cl(Int\lambda \wedge Cl\nu) \leq Cl(\lambda \wedge \nu)$. Therefore we obtain $Cl(\lambda \wedge \nu) \neq \Phi$ and hence $\lambda \wedge \nu \neq \Phi$. By Lemma 2, $x \in pCl\lambda$ and hence $Cl\lambda \leq pCl\lambda$. ■

We use Theorem 3(1) and Theorem 4, and prove:

Theorem 8 *For a fuzzy set ν in an fts X, the following hold:*
(1) *$\alpha Cl\nu = Cl\nu$, for every $\nu \in FSPO(X)$.*
(2) *$sCl\nu = IntCl\nu$, for every $\nu \in FPO(X)$.*

Proof. (1) Clearly $sCl\nu \leq Cl\nu$. Next, let $\nu \in FSPO(X)$. Then by Theorem 4, $\nu \leq ClIntCl\nu$ and by Theorem 3 (3), we have $\alpha Cl\nu \geq \nu \vee ClIntCl\nu = Cl\nu$. Hence $\alpha Cl\nu = Cl\nu$.

(2) Let $\nu \in FPO(X)$. Then $\nu \leq IntCl\nu$ and by Theorem 3(1), we have $sCl\nu = \nu \vee IntCl\nu = IntCl\nu$. ■

In view of Theorem 8, we have the following theorem, proof of which follows from Theorem 6:

Theorem 9 *The following are equivalent for a function $f : X \to Y$:*
(1) *f is f.a.c.S.*
(2) *$Clf^{-1}(\nu) \leq f^{-1}(\alpha Cl\nu)$, for every $\nu \in FSPO(Y)$.*
(3) *$Clf^{-1}(\nu) \leq f^{-1}(pCl\nu)$, for every $\nu \in FSO(Y)$.*
(4) *$f^{-1}(\nu) \leq Intf^{-1}(sCl\nu)$, for every $\nu \in FPO(Y)$.*

Definition 7 *[12] A function $f : X \to Y$ is said to be fuzzy almost open (resp. fuzzy almost closed) in Nanda's sense, briefly, f.a.o.N (resp. f.a.c.N.), if $f(\mu)$ is fuzzy open (resp. fuzzy closed) in Y, for each fuzzy regularly open (resp. fuzzy regularly closed) set μ in X.*

Theorem 10 *[2] Let $f : X \to Y$ be a fuzzy open and f.a.c.S. function. Then for each fuzzy open set ν in Y, $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$.*

We generalize Theorem 10 as:

Theorem 11 *A function $f : X \to Y$ is f.a.o.N and f.a.c.S. if and only if $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.*

Proof. ($\Rightarrow$) Let $\nu \in FSO(Y)$. Since f is f.a.c.S., by Theorem 6(3), $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$. Since f is fuzzy almost open, we have

$$f^{-1}(Cl\nu) = f^{-1}(ClInt\nu) \leq Clf^{-1}(Int\nu) = Clf^{-1}(\nu)$$

Therefore, we obtain $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.

($\Leftarrow$) It follows from Theorem 6(3) that f is fuzzy almost open. Let ψ be any fuzzy regularly closed set in Y. Then $\psi = ClInt\psi$ and hence $\psi \in FSO(Y)$. By the hypothesis, $Clf^{-1}(\psi) = f^{-1}(\psi)$ and hence $f^{-1}(\psi)$ is fuzzy closed in X. Therefore, by Theorem 5 f is f.a.c.S. ■

Next, we define:

Definition 8 *A surjective function $f : X \to Y$ is said to be fuzzy almost quasi-compact, if $f^{-1}(\lambda)$ is fuzzy regularly open in X implies λ is fuzzy open in Y.*

Then the following is immediate:

Theorem 12 *A bijective function $f : X \to Y$ is fuzzy almost quasi-compact if and only if the image of every fuzzy regularly open (resp. fuzzy regularly closed) inverse set is fuzzy open (resp. fuzzy closed).*

We use Theorem 12 and prove following characterization of fuzzy almost open functions:

Theorem 13 *If $f : X \to Y$ is a bijective function, then the following are equivalent:*
(1) *f is fuzzy almost open.*
(2) *f is fuzzy almost closed.*
(3) *f is fuzzy almost quasi-compact.*
(4) *f^{-1} is f.a.c.S.*

Proof. (1) $\Rightarrow$ (2) Let λ be a fuzzy regularly closed set in X. Then λ^c is fuzzy regularly open. Therefore $f(\lambda^c)$ is fuzzy open, or $(f(\lambda))^c$ is fuzzy open. This proves that $f(\lambda)$ is fuzzy closed and consequently f is fuzzy almost closed.

(2) $\Rightarrow$ (3) Let $f^{-1}(\psi)$ be fuzzy regularly closed. Then by Theorem 12, $ff^{-1}(\psi)$ is fuzzy closed, that is, ψ is fuzzy closed. This gives that f is fuzzy almost quasi-compact.

(3) $\Rightarrow$ (4) Let μ be a fuzzy regularly open set in X. Then $f^{-1}f(\mu) = \mu$ is fuzzy regularly open. Hence $f(\mu)$ is fuzzy open, that is, $(f^{-1})^{-1}(\mu)$ is fuzzy open implies f^{-1} is fuzzy almost continuous.

(4) $\Rightarrow$ (1) If λ is a fuzzy regularly open set in X. Then by hypothesis, $(f^{-1})^{-1}(\lambda)$ is a fuzzy open set of Y implies f is fuzzy almost open. ■

Theorem 14 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then we have:*
(1) *if a surjective function f is f.a.c.S. and gof is fuzzy open (resp. fuzzy closed), then g is fuzzy almost open (resp. fuzzy almost closed).*
(2) *if f is f.a.c.S. and gof is fuzzy quasi-compact, then g is fuzzy almost quasi-compact.*

Proof. (1) Suppose f is f.a.c.S. and gof is fuzzy open. Let ψ be any fuzzy regularly open set of Y. Since f is fuzzy almost continuous, therefore $f^{-1}(\psi)$ is a fuzzy open subset of X. Since, gof is fuzzy open, therefore $(gof)(f^{-1}(\psi))$ is a fuzzy open subset of Z, that is, $(gof)(f^{-1}(\psi)) = g(\psi)$ is fuzzy open. This proves that g is fuzzy almost open.

(2) Let f be f.a.c.S. and gof fuzzy quasi-compact. Let ψ be a fuzzy set in Z such that $g^{-1}(\psi)$ is fuzzy regularly open set in Y. Then by fuzzy almost-continuity of f, $f^{-1}g^{-1}(\psi)$ is fuzzy open in X. But $f^{-1}(g^{-1}(\psi)) = (gof)^{-1}(\psi)$ is fuzzy open, implies ψ is fuzzy open. This proves that g is fuzzy almost quasi-compact. ■

We generalize Theorem 7.10 [4] and obtain:

Theorem 15 *For each $i \in J$, let $f_i : X_i \to Y_i$ be functions where each X_i (resp. Y_i) is product related to X_j (resp. Y_j). Define $f : \Pi X_i \to \Pi Y_i$ by $f((x_i)) = (f_i(x_i))$. If f_i is f.a.c.S., for each $i \in J$, then f is f.a.c.S.*

Proof. Let $j \in J$ be fixed and $x_j \in X_j$. Let ν_j be a fuzzy regularly open set with $f_j(x_j) \in \nu_j$. Take $(x_i) \in \Pi X_i$ whose jth coordinate is x_j. Then the set $\nu = \Pi_{i\neq j} Y_i \times \ \nu_j$ is a fuzzy open set in ΠY_i with $(f_i(x_i)) \in \nu$ and $Cl\nu = \Pi_{i\neq j} Y_i \times \ Cl\nu_j$. Then

$$IntCl\nu = \Pi_{i\neq j} Y_i \times \ IntCl\nu_j = \Pi_{i\neq j} Y_i \times \ \nu_j = \nu,$$

which shows that ν is a fuzzy regularly open set. The fuzzy almost continuity of f gives the existence of a fuzzy open set μ with $(x_i) \in \mu$ such that $f(\mu) \leq$

$\Pi_{i\neq j}Y_i \times \ \nu_j = \nu$. Thus there exists a basic fuzzy open set $\gamma = \Pi Y_i \times \ \gamma_{i_1} \times \gamma_{i_2} \times \ldots \times \ \gamma_{i_j} \times \ldots \times \ \gamma_{i_n} \leq \mu$ with $(x_i) \in \gamma$ such that $f(\gamma) \leq \nu$, which implies $f_j(\gamma_j) \leq \nu_j$. This shows that f_j is fuzzy almost continuous and consequently f_i is fuzzy almost continuous, for each $i \in J$. ■

Theorem 16 *A mapping $f : X \to Y$ is f.a.c.S. at a fuzzy point $x \in X$, if for every fuzzy open neighborhood ν of $f(x)$, there is a fuzzy open neighborhood μ of x such that $f(\mu) \leq IntCl\nu$.*

Proof. Let x be a fuzzy point in X and let ν be a fuzzy open neighborhood of $f(x)$. Then there exists a fuzzy open set β such that $f(x) \in \beta \leq \nu$ and hence $f(x) \in IntCl\nu = \omega \in FRO(Y)$. Then there exists a fuzzy open set $\mu = f^{-1}(\omega)$ such that $x \in \mu$ and $f(\mu) \leq IntCl\nu$. ■

We use Theorem 16 and characterize f.a.c.S. functions in terms of fuzzy projections in product related spaces:

Theorem 17 *Let $\{Y_i : i \in J\}$ be a family of fts's, $f : X \to \Pi Y_i$ a function and $p_i : \prod Y_i \to Y_i$, the ith projection function. Then f is f.a.c.S. if and only if $p_i f$ is f.a.c.S., for each $i \in J$.*

Proof. Suppose f is f.a.c.S. Let $x \in X$ and ν_j a fuzzy regularly open set in Y_j such that $(p_j f)(x) = p_j((y_i)) \in \nu_j$, where $f(x) = (y_i) \in \Pi Y_i$. Then

$$\begin{aligned} IntClp_j^{-1}(\nu_j) &= Int(\Pi Y_i \times \ Cl\nu_j) \\ &= \Pi Y_i \times \ IntCl\nu_j = \Pi Y_i \times \ \nu_j = p_j^{-1}(\nu_j) \end{aligned}$$

Therefore $p_j^{-1}(\nu_j)$ is fuzzy regularly open in ΠY_i and $f(x) = (y_i) \in p_j^{-1}(\nu_j)$. Then by Theorem 16 f is f.a.c.S. implies the existence of a fuzzy open set μ in X with $x \in \mu$ such that $f(\mu) \leq p_j^{-1}(\nu_j)$. Consequently, $(p_j f)(\mu) = p_j(f(\mu)) \leq p_j(p_j^{-1}(\nu_j)) = \nu_j$, implies $p_j f$ is f.a.c.S. It follows that $p_i f$ is f.a.c.S. for each $i \in J$.

Conversely, suppose $p_i f$ is f.a.c.S. for each $i \in J$. Let $x \in X$ and ν a fuzzy regularly open set with $f(x) = (y_i) \in \nu$. Then there exists a basic fuzzy open set $\Pi_{i\neq i_k}Y_i \times \ \nu_{i_1} \times \ \nu_{i_2} \times \ldots \times \ \nu_{i_n} \leq \nu$ which contains $f(x)$. Thus $f(x) \in \Pi_{i\neq i_k}Y_i \times \ IntCl\nu_{i_1} \times \ IntCl\nu_{i_2} \times \ldots \times \ IntCl\nu_{i_n} \leq IntCl\nu = \nu$. Since each $IntCl\nu_{i_j}$ is fuzzy regularly open in Y_{i_j} and $p_{i_j} f$ is f.a.c.S., then by Theorem 16 there exist fuzzy open sets μ_i in X with $x \in \mu_i$ such that $(p_{i_j} f)(\mu_i) \leq IntCl\mu_{i_j}$. Let $\eta = \wedge_{i=1}^{n}\mu_i$. Then $f(\eta) \leq \Pi_{i\neq i_j}Y_i \times IntCl\nu_{i_1} \times \ldots \times IntCl\nu_{i_n} \leq IntCl\nu = \nu$. This shows f is f.a.c.S. ■

In view of Theorem 17, the following is immediate:

Theorem 18 *Let Y_i, $i \in J$ be the fts's and let $f_i : X \to Y_i$, $i \in J$, be a family of functions. Define $f : X \to \Pi Y_i$ by $f(x) = (f_i(x))$. Then f is f.a.c.S. if and only if each f_i is f.a.c.S.*

4 Fuzzy Almost Continuous Functions in Hussain's Sense

Definition 9 *[2] A function $f : X \to Y$ is said to be a fuzzy almost continuous function in the sense of Hussain (briefly, f.a.c.H.) at $x_\alpha \in X$, if for each fuzzy open set ν in Y with $f(x_\alpha) \in \nu$, $Clf^{-1}(\nu)$ is a fuzzy neighborhood of x_α. If f is f.a.c.H. at each point of X, then f is called f.a.c.H.*

Definition 10 *A fuzzy set λ is said to be fuzzy dense in another fuzzy set μ, both being fuzzy sets in an fts X, if $Cl\lambda = \mu$.*

Then the following is immediate:

Theorem 19 *A function $f : X \to Y$ is f.a.c.H. at $x_\alpha \in X$ if and only if for each fuzzy open set ν in Y with $f(x_\alpha) \in \nu$, there exists a fuzzy open set μ in X with $x_\alpha \in \mu$, such that $f^{-1}(\nu)$ is fuzzy dense in μ.*

Lemma 3 *[4] Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. If λ is a fuzzy set of X and μ is a fuzzy set in Y, then $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Using Theorem 19 and Lemma 3, we have:

Theorem 20 *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$, be the graph of f. Then g is f.a.c.H. if and only if f is f.a.c.H.*

Proof. ($\Rightarrow$) Let g be f.a.c.H. and $x_\alpha \in X$. Consider a fuzzy open set ν in Y with $f(x_\alpha) \in \nu$. Then $1_X \times \nu$ is a fuzzy open set in $1_X \times 1_Y$ with $(x_\alpha, f(x_\alpha)) \in 1_X \times \nu$. Since g is f.a.c.H., therefore by Lemma 3, $Clg^{-1}(1_X \times \nu) = Cl(1_X \wedge f^{-1}(\nu)) = Clf^{-1}(\nu)$ is a fuzzy neighborhood of x_α. Thus f is f.a.c.H. at the point x_α.

($\Leftarrow$) Let f be f.a.c.H. We show that g is a f.a.c.H. at the point $x_\alpha \in X$. Let ω be a fuzzy open set in $1_X \times 1_Y$ such that $g(x_\alpha) = (x_\alpha, f(x_\alpha)) \in \omega$. Then by Theorem 19, there exist fuzzy open sets μ in X and ν in Y such that $x_\alpha \in \mu$, $f(x_\alpha) \in \nu$ and $\mu \times \nu \leq \omega$. By Lemma 3, $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu)$. Since f is f.a.c.H., by Theorem 19, there exists a fuzzy open set ψ in X with $x_\alpha \in \psi$ such that $\psi \leq \mu$ and $f^{-1}(\nu)$ is fuzzy dense in ψ. Thus, $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu) \geq \psi \wedge f^{-1}(\nu)$ gives $Cl\ g^{-1}(\omega) \geq Clg^{-1}(\mu \times \nu) \geq Cl(\psi \wedge f^{-1}(\nu)) \geq \psi$. Thus $Clg^{-1}(\omega)$ is a fuzzy neighborhood of x_α implies g is f.a.c.H. at the point x_α. ■

The concept of almost weakly continuous functions has been defined and studied by D. S. Jankovic [8]. In fuzzy settings, we define this as:

Definition 11 *A function $f : X \to Y$ is said to be fuzzy almost weakly continuous (briefly, f.a.w.c) $f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$, for every fuzzy open set ν in Y.*

The notion of fuzzy almost weakly continuous function is a weaker notion than that of f.a.c.H. as proved in Theorem 22.

Theorem 21 *For a function $f : X \to Y$, the following are equivalent:*
(1) f is f.a.w.c.
(2) $ClIntf^{-1}(\nu) \leq f^{-1}(Cl\nu)$ for every fuzzy open set ν in Y.
(3) For each $x_\alpha \in X$ and each fuzzy open set ν such that $f(x_\alpha) \in \nu$, $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x_α.

Proof. (1) $\Rightarrow$ (2) Let ν be a fuzzy open set in Y. Then $(Cl\nu)^c$ is fuzzy open in Y and we have

$$\begin{aligned}\left(f^{-1}(Cl\nu)\right)^c &= f^{-1}((Cl\nu)^c) \\ &\leq IntClf^{-1}(Cl(Cl\nu)^c) \leq \left(ClIntf^{-1}(\nu)\right)^c.\end{aligned}$$

Therefore we obtain $ClIntf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.

(2) $\Rightarrow$ (3) Let $x \in X$ and ν be a fuzzy open set such that $f(x) \in \nu$. Since $(Cl\nu)^c$ is fuzzy open in Y, we have

$$\begin{aligned}\left(IntClf^{-1}(Cl\nu)\right)^c &= ClIntf^{-1}((Cl\nu)^c) \leq f^{-1}(Cl(Cl\nu)^c) \\ &= f^{-1}((IntCl\nu)^c) \leq f^{-1}(\nu^c) = \left(f^{-1}(\nu)\right)^c.\end{aligned}$$

Therefore, we obtain $x \in f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$ and hence $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x.
(3) $\Rightarrow$ (1) Let ν be a fuzzy open set in Y and $x \in f^{-1}(\nu)$. Then $f(x) \in \nu$ and $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x. Therefore, $x \in IntClf^{-1}(Cl\nu)$ and we obtain $f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$. ■

The following is immediate from Theorem 21:

Theorem 22 *Every f.a.c.H. function $f : X \to Y$ is f.a.w.c. function.*

Finally, we define:

Definition 12 *A function $f : X \to Y$ is said to be fuzzy nearly almost open, if there exists a fuzzy open basis $\mathcal{B}$ for the fuzzy topology on Y such that $f^{-1}(Cl\nu) \leq Clf^{-1}(\nu)$, for every $\nu \in \mathcal{B}$.*

Using fuzzy nearly almost open function, we give the partial converse of Theorem 22 as:

Theorem 23 *If a function $f : X \to Y$ is fuzzy nearly almost open and fuzzy almost weakly continuous, then f is f.a.c.H.*

Proof. Since f is fuzzy nearly almost open, there exists a fuzzy open basis $\mathcal{B}$ for the fuzzy topology on Y such that $f^{-1}(Cl\nu) \leq Clf^{-1}(\nu)$ for every $\nu \in \mathcal{B}$. Let ω be any fuzzy open set of Y. There exists a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $\omega = \vee\{\nu|\nu \in \mathcal{B}_0\}$. Therefore, we obtain

$$\begin{aligned} f^{-1}(\omega) &= f^{-1}(\vee_{\nu\in\mathcal{B}_0}\nu) \\ &= \vee_{\nu\in\mathcal{B}_0} f^{-1}(\nu) \leq \vee_{\nu\in\mathcal{B}_0} IntClf^{-1}(Cl\nu) \\ &\leq \vee_{\nu\in\mathcal{B}_0} IntClf^{-1}(\nu) \leq IntCl\ \left(\vee_{\nu\in\mathcal{B}_0} f^{-1}(\nu)\right) \\ &= IntCl\ f^{-1}(\omega) \end{aligned}$$

This shows that $f^{-1}(\omega) \in FPO(X)$ and hence f is f.a.c.H. ■

Combining Theorems 22 and 23, we have:

Theorem 24 *Let $f : X \to Y$ be a fuzzy nearly almost open function. Then f is f.a.w.c. if and only if f is f.a.c.S.*

5 Fuzzy Almost Continuous Functions in Both Senses

In the following theorems, we interconnect the independent notions of f.a.c.S. and f.a.c.H. functions and finally prove their equivalence under certain conditions.

Theorem 25 *[2] Let $f : X \to Y$ be a fuzzy open and f.a.c.S. function. Then f is f.a.c.H.*

Theorem 26 *[2] Let $f : X \to Y$ be a f.a.c.S. function. Then for each fuzzy open set ν in Y, $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.*

Theorem 27 *[1] If $f : X \to Y$ is a fuzzy weakly continuous and a fuzzy open function, then f is f.a.c.S.*

We use Theorem 27 and prove the partial converse of Theorem 25 as well as Theorem 26:

Theorem 28 *Let $f : X \to Y$ be a fuzzy open and f.a.c.H. function. If for every fuzzy open set ν in Y, $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, then f is f.a.c.S.*

Proof. Let f be f.a.c.H. and fuzzy open such that $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every fuzzy open set ν in Y. We show that f is f.a.c.S. Let $x_\alpha \in X$ such that $f(x_\alpha) \in \nu$. Since f is f.a.c.H., there exists a fuzzy open set μ in X such that $x_\alpha \in \mu \leq Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$. It follows that $f(\mu) \leq f(f^{-1}(Cl\nu)) \leq Cl\nu$, f is fuzzy weakly continuous. Since f is fuzzy open, then by Theorem 27, that is, f is f.a.c.S. ■

Combining Theorems 25 and 28 we have the following theorem, which improves the main result of [2] (Theorem 5.5):

Theorem 29 *Let $f : X \to Y$ be a fuzzy open function satisfying $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every fuzzy open ν in Y. Then f is f.a.c.H. if and only if f is f.a.c.S.*

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Received: March 18, 2008