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On the Difference Equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}$$

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Abstract

We study the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, ...$$

where initial values are non zero real numbers with $x_{-5}x_{-2} \neq 1,\ x_{-4}x_{-1} \neq 1,\ x_{-3}x_0 \neq 1$.

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1 Introduction

In this paper a solution of the following difference equation was investigated

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, ... \tag{1}$$

where the initial conditions are arbitrary non zero real numbers with $x_{-5}x_{-2} \neq 1,\ x_{-4}x_{-1} \neq 1,\ x_{-3}x_0 \neq 1$.

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [1-9]. Cinar [2] investigated the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}.$$

Also, In [3], Cinar gave that the solution of the difference equation

$$x_{n+1}=\frac{x_{n-1}}{-1+x_nx_{n-1}}.$$

Karatas et al.[8] gave that the solution of the difference equation

$$x_{n+1}=\frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$$

Aloqeili [1] has obtained the solutions of the difference equation

$$x_{n+1}=\frac{x_{n-1}}{a-x_nx_{n-1}},$$

Elabbasy et al. [5] investigated the global stability, periodicity character and give the solution of special case of the following recursive sequence

$$x_{n+1}=ax_n-\frac{bx_n}{cx_n-dx_{n-1}}.$$

Elabbasy et al. [7] investigated the global stability, periodicity character and give the solution of some special cases of the difference equation

$$x_{n+1}=\frac{dx_{n-l}x_{n-k}}{cx_{n-s}-b}+a.$$

2 Main Results

In this section we give a specific form of the solutions of Eq.(1)

Theorem 1 *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(1). Then for $n=0,1,...$*

$$\begin{aligned} x_{6n-5}&=\frac{f}{(-1+cf)^n}, \qquad x_{6n-4}=\frac{e}{(-1+be)^n},\\ x_{6n-3}&=\frac{d}{(-1+ad)^n}, \qquad x_{6n-2}=c\left(-1+cf\right)^n,\\ x_{6n-1}&=b\left(-1+be\right)^n, \qquad x_{6n}=a\left(-1+ad\right)^n, \end{aligned} \tag{2}$$

where $x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a.$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned} x_{6n-11}&=\frac{f}{(-1+cf)^{n-1}}, \qquad x_{6n-10}=\frac{e}{(-1+be)^{n-1}},\\ x_{6n-9}&=\frac{d}{(-1+ad)^{n-1}}, \qquad x_{6n-8}=c\left(-1+cf\right)^{n-1},\\ x_{6n-7}&=b\left(-1+be\right)^{n-1}, \qquad x_{6n-6}=a\left(-1+ad\right)^{n-1}, \end{aligned}$$

Now, it follows from Eq.(9) that

$$\begin{aligned} x_{6n-5} &= \frac{x_{6n-11}}{-1+x_{6n-8}x_{6n-11}} = \frac{\dfrac{f}{(-1+cf)^{n-1}}}{-1+\dfrac{f}{(-1+cf)^{n-1}}c\,(-1+cf)^{n-1}} \\ &= \frac{f}{(-1+cf)^{n-1}\,(-1+cf)}. \end{aligned}$$

Hence, we have

$$x_{6n-5} = \frac{f}{(-1+cf)^{n}}.$$

Similarly

$$\begin{aligned} x_{6n-2} &= \frac{x_{6n-8}}{-1+x_{6n-5}x_{6n-8}} = \frac{c\,(-1+cf)^{n-1}}{-1+\dfrac{f}{(-1+cf)^{n}}c\,(-1+cf)^{n-1}} \\ &= \frac{c\,(-1+cf)^{n-1}}{-1+\dfrac{f}{(-1+cf)}c} = \frac{c\,(-1+cf)^{n}}{1-cf+cf}. \end{aligned}$$

Hence, we have

$$x_{6n-2} = c\,(-1+cf)^{n}\,.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2 *Eq.(1) has three equilibrium points which are* $0, \sqrt{2}, -\sqrt{2}$.

Proof: For the equilibrium points of Eq.(1), we can write

$$\overline{x} = \frac{\overline{x}}{-1+\overline{x}^2}.$$

Then we have

$$-\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^2 - 2) = 0.$$

Thus the equilibrium points of Eq.(1) are $0, \sqrt{2}, -\sqrt{2}$.

Theorem 3 *Eq.(1) has a periodic solutions of period six iff* $cf = be = ad = 2$ *and will be take the form* $\{f, e, d, c, b, a, f, e, d, c, b, a, ...\}$.

Proof: First suppose that there exists a prime period six solution

$$f, e, d, c, b, a, f, e, d, c, b, a, ...$$

of Eq.(1), we see from Eq.(2) that

$$\begin{aligned} f &= \frac{f}{(-1+cf)^n}, \qquad e = \frac{e}{(-1+be)^n}, \\ d &= \frac{d}{(-1+ad)^n}, \qquad c = c\,(-1+cf)^n\,, \\ b &= b\,(-1+be)^n\,, \qquad a = a\,(-1+ad)^n\,, \end{aligned}$$

or,

$$\begin{aligned} (-1+cf)^{n-1} &= 1, \qquad (-1+be)^{n-1} = 1, \\ (-1+ad)^{n-1} &= 1. \end{aligned}$$

Then

$$cf = 2, \quad be = 2, \quad ad = 2.$$

Second suppose $cf = 2,\ be = 2,\ ad = 2$. Then we see from Eq.(2) that

$$\begin{aligned} x_{6n-5} &= f, \qquad x_{6n-4} = e, \\ x_{6n-3} &= d, \qquad x_{6n-2} = c, \\ x_{6n-1} &= b, \qquad x_{6n} = a. \end{aligned}$$

Thus we have a period six solution and the proof is complete.
Lemma 1. Eq.(1) has no prime period two solution.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (1).
Example 1. We consider $x_{-5} = 0.7,\ x_{-4} = 0.3,\ x_{-3} = 0.5,\ x_{-2} = 0.4,\ x_{-1} = 0.8,\ x_0 = 0.2$ See Fig. 1.

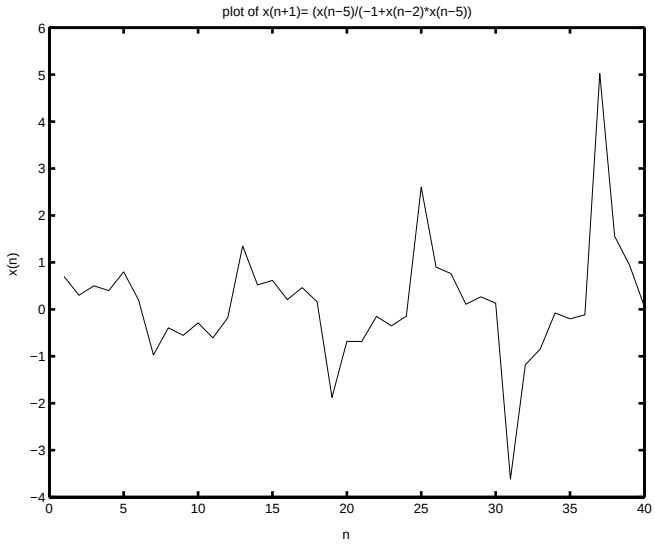

Figure 1:

Example 2. See Fig. 2, since $x_{-5} = 0.8,\ x_{-4} = 0.3,\ x_{-3} = 0.7,\ x_{-2} = 4,\ x_{-1} = 5,\ x_0 = 7$.

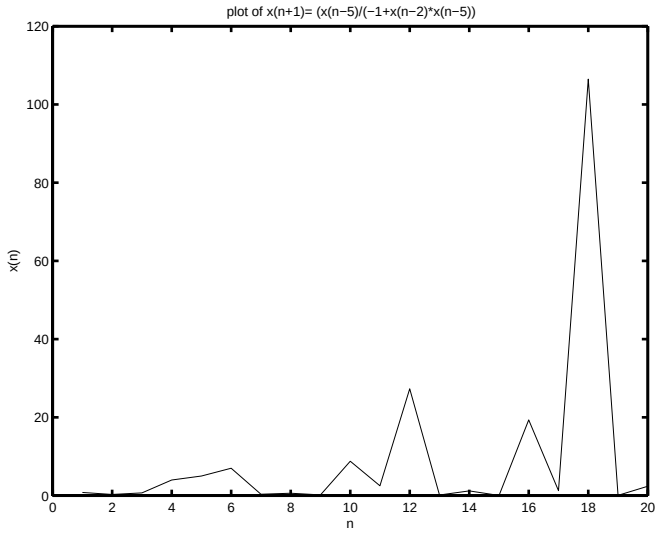

Figure 2:

Example 3. In Fig. 3, we assume $x_{-5} = 4,\ x_{-4} = 0.25,\ x_{-3} = 5,\ x_{-2} = 0.5,\ x_{-1} = 8,\ x_0 = 0.4$.

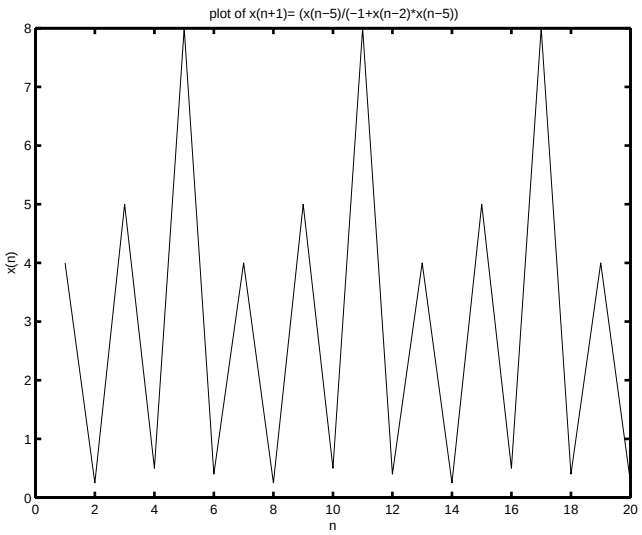

Figure 3:

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