**Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 35, 1725 - 1731**

# A Note on Quasi-Ideals in Regular Γ-Semirings[1]

**Ronnason Chinram**

Department of Mathematics, Faculty of Science,
Prince of Songkla University, Hat Yai, Songkhla 90112, Thailand
e-mail:ronnason.c@psu.ac.th

**Abstract**

The notions of quasi-ideals for rings and semigroups were introduced by O. Steinfeld in 1953 and 1956, respectively. The notions of quasi-ideals for Γ-semigroups, semirings and Γ-semirings were defined analogously. In [4], the author studied some properties of quasi-ideals of Γ-semirings. In this paper, some properties of quasi-ideals of regular Γ-semirings are provided.

**Mathematics Subject Classification:** 20M10, 16Y99

**Keywords:** Γ-semirings, quasi-ideals, regular Γ-semirings

## 1 Introduction

The notions of quasi-ideals for rings and semigroups were introduced by O. Steinfeld in [13] and [14], respectively. We can see general properties of quasi-ideals for rings and semigroups in [15]. The notion of a Γ-semigroup was introduced by M. K. Sen in [10]. Γ-semigroups generalize semigroups. Many classical notions of semigroups have been extended to Γ-semigroups ([1], [2], [3], [5], [8], [10] and [11]). The notions of quasi-ideals for Γ-semigroups, semirings and Γ-semirings were defined analogously. In [12], M. Shabir, A. Ali and S. Batool studied some properties of quasi-ideals of semirings and in [3], the author studied some properties of quasi-ideals of Γ-semigroups. In [4], the author studied some properties of quasi-ideals of Γ-semirings.

In this paper, some properties of quasi-ideals of regular Γ-semirings are provided.

[1]This research was supported by the Commission on Higher Education and a grant of Thailand Research Fund (TRF) - MRG5080220.

## 2 Preliminaries

Let $(S,+)$ be a commutative semigroup and $\Gamma$ a nonempty set. $S$ is called a $\Gamma$-*semiring* if $S$ is a $\Gamma$-semigroup (that is $S$ satisfies the identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a,b,c \in S$ and $\gamma,\mu \in \Gamma$) and for all $a,b,c \in S$ and $\gamma \in \Gamma$,

$$a\gamma(b+c) = a\gamma b + a\gamma c \text{ and } (b+c)\gamma a = b\gamma a + c\gamma a.$$

**Example 2.1.** Let $S$ be an arbitrary semiring and $\Gamma$ an arbitrary nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a,b \in S$ and $\gamma \in \Gamma$. It is easy to see that $S$ is a $\Gamma$-semiring. Thus a semiring can be considered to be a $\Gamma$-semiring. Since every ring is a semiring, a ring can be considered to be a $\Gamma$-semiring. ■

**Example 2.2.** Let $S$ be a $\Gamma$-semiring and $\alpha$ a fixed element in $\Gamma$. We define $a \cdot b = a\alpha b$ for all $a,b \in S$. We can show that $(S,+,\cdot)$ is a semiring. ■

In this paper, we shall assume that $S$ has a $\Gamma$-absorbing zero 0, that is $a+0 = a = 0+a$ and $0\gamma a = 0 = a\gamma 0$ for all $a \in S$ and $\gamma \in \Gamma$. A nonempty subset $T$ of a $\Gamma$-semiring $S$ is called a *sub*$\Gamma$-*semiring* of $S$ if $T$ is a subsemigroup of $(S,+)$ and $a\gamma b \in T$ for all $a,b \in T$ and $\gamma \in \Gamma$.

**Example 2.3.** Let $S$ be a semiring and $T$ a subsemiring of $S$. Then $T$ is a subsemigroup of $(S,+)$ and $ab \in T$ for all $a,b \in T$. Let $\Gamma$ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a,b \in S$ and $\gamma \in \Gamma$. By Example 2.1, $S$ is a $\Gamma$-semiring. We have for all $a,b \in T, a\gamma b = ab \in T$. Then $T$ can be considered to be a sub$\Gamma$-semiring of a $\Gamma$-semiring $S$.

Let $R$ be a ring and $T$ a subring of $R$. Similarly, $R$ can be considered to be a $\Gamma$-semiring and $T$ can be considered to be a sub$\Gamma$-semiring of a $\Gamma$-semiring $S$. ■

A nonempty subset $T$ of a $\Gamma$-semiring $S$ is called a *left (resp. right) ideal* of $S$ if $T$ is a subsemigroup of $(S,+)$ and $x\gamma a \in T$ (resp. $a\gamma x \in T$) for all $a \in T, \gamma \in \Gamma$ and $x \in S$. If $T$ is both a left ideal and a right ideal of $S$, then $T$ is called an *ideal* of $S$. It is easy to see that every left ideal, right ideal and ideal of $S$ is a sub$\Gamma$-semiring of $S$.

Let $A$ and $B$ be two nonempty subsets of a $\Gamma$-semiring $S$ and $\mathbb{N}$ the set of all natural numbers. Let

$$A+B = \{a+b \mid a \in A \text{ and } b \in B\},$$

$$A\Gamma B \text{ denotes the set of all finite sums of the form } \sum a_i\gamma_i b_i$$

$$\text{where } a_i \in A, \gamma_i \in \Gamma \text{ and } b_i \in B,$$

$\mathbb{N}A$ denotes the set of all finite sums of the form $\sum n_i a_i$ where $n_i \in \mathbb{N}$ and $a_i \in B$.

Let $S$ be a $\Gamma$-semiring. By a *quasi-ideal* $Q$ we mean a subsemigroup $Q$ of $(S, +)$ such that $S\Gamma Q \cap Q\Gamma S \subseteq Q$. It is clear that every left ideal and right ideal of a $\Gamma$-semiring $S$ is a quasi-ideal of $S$. Moreover, each quasi-ideal of $S$ is a sub$\Gamma$-semiring of $S$. In fact, $Q\Gamma Q \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$.

**Example 2.4.** Let $S$ be a semiring and $T$ a left ideal (resp. right ideal, ideal, quasi-ideal) of $S$. Similar to Example 2.3, $T$ can be considered to be a left ideal (resp. right ideal, ideal, quasi-ideal) of a $\Gamma$-semiring $S$.

Let $R$ be a ring and $T$ a left ideal (resp. right ideal, ideal, quasi-ideal) of $R$. Similarly, $T$ can be considered to be a left ideal (resp. right ideal, ideal, quasi-ideal) of a $\Gamma$-semiring $R$. ■

Let $X$ be a nonempty subset of a $\Gamma$-semiring $S$. By the term left ideal $(X)_l$ (resp. right ideal $(X)_r$, ideal $(X)_i$, quasi-ideal $(X)_q$) of $S$ generated by $X$, we mean the smallest left ideal (resp. right ideal, ideal, quasi-ideal) of $S$ containing $X$, that is the intersection of all left ideals (resp. right ideals, ideals, quasi-ideals) of $S$ containing $X$. We have that

**Theorem 2.1.** *[4] Let $S$ be a $\Gamma$-semiring and $X$ a nonempty subset of $S$. Then*
*(i)* $(X)_l = \mathbb{N}X + S\Gamma X$,
*(ii)* $(X)_r = \mathbb{N}X + X\Gamma S$,
*(iii)* $(X)_i = \mathbb{N}X + S\Gamma X + X\Gamma S + S\Gamma X\Gamma S$,
*(iv)* $(X)_q = \mathbb{N}X + (S\Gamma X \cap X\Gamma S)$.

Let $S$ be a $\Gamma$-semiring. An element $a \in S$ is called a *left identity (resp. right ideal)* of $S$ if $x = a\gamma x$ (resp. $x = x\gamma a$) for all $x \in S$ and $\gamma \in \Gamma$. If $a$ is both a left and right identity, then $a$ is called an *identity* of $S$. The following corollary holds.

**Corollary 2.2.** *Let $S$ be a $\Gamma$-semiring with an identity and $X$ a nonempty subset of $S$. Then*
*(i)* $(X)_l = S\Gamma X$,
*(ii)* $(X)_r = X\Gamma S$,
*(iii)* $(X)_i = S\Gamma X\Gamma S$,
*(iv)* $(X)_q = S\Gamma X \cap X\Gamma S$.

The following theorem is similar to the case of semigroups, $\Gamma$-semigroups, rings and semirings.

**Theorem 2.3.** *[4] The intersection of a left ideal $L$ and a right ideal $R$ of a $\Gamma$-semiring $S$ is a quasi-ideal of $S$.*

In case of semigroups and $\Gamma$-semigroups it is true that every quasi-ideal can be written as an intersection of a left ideal and a right ideal ([3] and [15]). However, this result has no analogue for rings ([6], [7], [9] and [15]), by Example 2.3 and Example 2.4, this implies that this result has no analogue for semirings and $\Gamma$-semirings.

An element $a$ of $S$ is *regular* if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A $\Gamma$-semiring $S$ is *regular* if every element in $S$ is regular. The following theorem is true.

**Theorem 2.4.** *[4] Every quasi-ideal of a regular $\Gamma$-semiring $S$ can be written in the form $Q = R \cap L$ for some a right ideal $R$ and a left ideal $L$ of $S$.*

In this paper, some properties of quasi-ideals of regular $\Gamma$-semirings are provided.

# 3 Main Results

The following theorem holds.

**Theorem 3.1.** *Let $S$ be a $\Gamma$-semiring. If $S$ is regular, then $R\Gamma L = R \cap L$ for all right ideal $R$ and left ideal $L$ of $S$. The converse is true if $S$ has an identity.*

**Proof.** Let $S$ be a regular $\Gamma$-semiring and $R$ and $L$ be a right ideal and a left ideal of $S$. So $R\Gamma L \subseteq R\Gamma S \cap S\Gamma L \subseteq R \cap L$. Next, let $a \in R \cap L$. Since $S$ is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Since $a \in L, x\beta a \in L$. So $a = a\alpha x\beta a \in R\Gamma L$.

Conversely, assume $R\Gamma L = R \cap L$ for all right ideal $R$ and left ideal $L$ of $S$. Let $x \in S$ and $\gamma$ a fixed element in $\Gamma$. Then $x\gamma S^1$ and $S^1\gamma x$ is a right ideal and a left ideal of $S$, respectively. By assumption, $(x\gamma S^1)\Gamma(S^1\gamma x) = x\gamma S^1 \cap S^1\gamma a$. So $x \in (x\gamma S^1)\Gamma(S^1\gamma x)$. Then $x = \sum a_i\alpha_i b_i$ for some $a_i \in x\gamma S^1, b_i \in S^1\gamma x$ and $\alpha_i \in \Gamma$. Since $S$ has an identity, for each $i$, we have $a_i = x\gamma s_i$ and $b_i = t_i\gamma x$ for some $t_i, s_i \in S$. Then $x = \sum a_i\alpha_i b_i = \sum(x\gamma s_i)\alpha_i(t_i\gamma x) = \sum x\gamma(s_i\alpha_i t_i)\gamma x = x\gamma(\sum s_i\alpha_i t_i)\gamma x$. Therefore $x$ is regular. ■

**Theorem 3.2.** *Let $S$ be a $\Gamma$-semiring and $a \in S$. If $a$ is regular, then $(a)_r\Gamma(a)_l = (a)_r \cap (a)_l$. The converse is true if $S$ has an identity.*

**Proof.** Assume $a$ is regular. We have $(a)_r\Gamma(a)_l \subseteq ((a)_r\Gamma S) \cap (S\Gamma(a)_l) \subseteq (a)_r \cap (a)_l$. Let $x \in (a)_r \cap (a)_l$. Then $x \in (a)_r$ and $x \in (a)_l$. Since $x \in (a)_l$, $x = a$ or $x = s\gamma a$ for some $\gamma \in \Gamma$ and $s \in S$.

*Case 1* : $x = a$. Then $x$ is regular. Thus $x = x\alpha y\beta x$ for some $y \in S$ and $\alpha, \beta \in \Gamma$. So $x = x\alpha y\beta x = a\alpha(y\beta a) \in (a)_r\Gamma(a)_l$.

*Case 2* : $x = s\gamma a$. Since $a$ is regular, there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha y\beta a$. So $x = s\gamma a = s\gamma(a\alpha y\beta a) = (s\gamma a)\alpha y\beta a = x\alpha(y\beta a) \in (a)_r\Gamma(a)_l$.

Therefore, $(a)_r\Gamma(a)_l = (a)_r \cap (a)_l$.

Conversely, assume $(a)_r\Gamma(a)_l = (a)_r \cap (a)_l$. Then $a \in (a)_r\Gamma(a)_l$. Then $a \in (a\Gamma S)\Gamma(S\Gamma a) \subseteq a\Gamma S\Gamma a$. Then $a = \sum a\alpha_i s_i\beta_i a$ for some $s_i \in S$ and $\alpha_i, \beta_i \in \Gamma$. Since $S$ has an identity 1, $a = \sum a\alpha_i s_i\beta_i a = \sum a\gamma 1\alpha_i s_i\beta_i 1\gamma a = a\gamma(\sum 1\alpha_i s_i\beta_i 1)\gamma a$ for a fixed element $\gamma \in \Gamma$. Therefore, $a$ is regular. ■

The following corollary is obtained by Theorem 3.1 and Theorem 2.4.

**Corollary 3.3.** *Let $S$ be a $\Gamma$-semiring. Then every quasi-ideal $Q$ of $S$ can be written in the form $Q = R \cap L = R\Gamma L$ for some right ideal $R$ and left ideal $L$ of $S$.*

A nonempty subset $B$ of a $\Gamma$-semiring $S$ is called a *bi-ideal* of $S$ if $B$ is a sub$\Gamma$-semiring of $S$ and $B\Gamma S\Gamma B \subseteq B$. The following theorem holds.

**Theorem 3.4.** *Every quasi-ideal of a $\Gamma$-semiring $S$ is a bi-ideal of $S$.*

**Proof.** Let $Q$ be a quasi-ideal of $S$. Then $Q$ is a sub$\Gamma$-semiring of $S$ and $Q\Gamma S\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q \subseteq Q$. Hence $Q$ is a bi-ideal of $S$. ■

The converse of the above theorem is not generally true, for example we can see Example 8 in the case of semirings [12] because a semiring can be considered to be a $\Gamma$-semiring. The following theorem is similar to the case of rings and semirings.

**Theorem 3.5.** *Let $S$ be a regular $\Gamma$-semiring. Then the set of bi-ideals of $S$ coincides with the set of quasi-ideals of $S$.*

**Proof.** By Theorem 3.4, we have known every quasi-ideal of $S$ is a bi-ideal of $S$. Now, we show that a bi-ideal of $S$ is a quasi-ideal of $S$. Let $B$ be a bi-ideal of $S$. Let $a \in B\Gamma S \cap S\Gamma B$. Since $S$ is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. So $a \in a\Gamma S\Gamma a \subseteq (B\Gamma S)\Gamma S\Gamma(S\Gamma B) \subseteq B\Gamma S\Gamma B \subseteq B$. Hence $B$ is a quasi-ideal of $S$. The theorem is proved. ■

A $\Gamma$-semiring is called a *duo-$\Gamma$-semiring* if every one-sided (right or left) ideal of $S$ is an ideal of $S$. The following theorem holds.

**Theorem 3.6.** *Let $S$ be a duo-$\Gamma$-semiring. If $S$ is regular, then $Q_1\Gamma Q_2 = Q_1 \cap Q_2$ for all any two quasi-ideals $Q_1$ and $Q_2$ of $S$. The converse is true if $S$ has an identity.*

**Proof.** Assume $S$ is a regular duo-$\Gamma$-semiring. Let $Q_1$ and $Q_2$ be quasi-ideals of $S$. By Theorem 2.4, $Q_1$ and $Q_2$ can be written in the forms

$$Q_1 = R_1 \cap L_1 \text{ and } Q_2 = R_2 \cap L_2$$

for some $R_1, R_2$ and $L_1, L_2$ are suitable right ideals and left ideals of $S$, respectively. Since $S$ is a duo-$\Gamma$-semiring, $R_1, R_2, L_1$ and $L_2$ are ideals of $S$. Then $Q_1$ and $Q_2$ are ideals of $S$. By Theorem 3.1, we have $Q_1\Gamma Q_2 = Q_1 \cap Q_2$.

Conversely, assume $Q_1\Gamma Q_2 = Q_1 \cap Q_2$ for all any two quasi-ideals $Q_1$ and $Q_2$ of $S$. Let $x \in S$ and $\gamma$ a fixed element in $\Gamma$. Then $x\gamma S^1 \cap S^1\gamma x$ is a quasi-ideal of $S$. By assumption, $(x\gamma S^1 \cap S^1\gamma x)\Gamma(x\gamma S^1 \cap S^1\gamma x) = x\gamma S^1 \cap S^1\gamma x$. So $x \in (x\gamma S^1 \cap S^1\gamma x)\Gamma(x\gamma S^1 \cap S^1\gamma x)$. Then $x = \sum a_i\alpha_i b_i$ for some $a_i, b_i \in x\gamma S^1 \cap S^1\gamma x$ and $\alpha_i \in \Gamma$. Since $S$ has an identity, for each $i$, we have $a_i = x\gamma s_i$ and $b_i = t_i\gamma x$ for some $t_i, s_i \in S$. Then $x = \sum a_i\gamma_i b_i = \sum(x\gamma s_i)\alpha_i(t_i\gamma x) = \sum x\gamma(s_i\alpha_i t_i)\gamma x = x\gamma(\sum s_i\alpha_i t_i)\gamma x$. Therefore $x$ is regular. ∎

An element $a$ in a $\Gamma$-semiring $S$ is called a *duo element* if the principal right ideal $(a)_r$ and the principal left ideal $(a)_l$ of $S$ generated by $a$ are equal. We have that

**Theorem 3.7.** *Let $S$ be a $\Gamma$-semiring. Then $S$ is a duo-$\Gamma$-semiring if and only if every element of $A$ is a duo element.*

**Proof.** If $S$ is a duo-$\Gamma$-semiring, then evidently every element of $S$ is duo.

Conversely, assume that every element of $S$ is duo. Let $R$ be an arbitrary right ideal of $S$. Let $x \in S, \gamma \in \Gamma$ and $a \in R$. Then $x\gamma a \in (a)_l = (a)_r \subseteq R$. Hence $R$ is a left ideal of $S$.

Similarly, we can show that any left ideal of $S$ is a right ideal of $S$. ∎

**ACKNOWLEDGEMENTS.** The author is grateful to the Commission on Higher Education and the Thailand Research Fund (TRF) for grant support.

# References

[1] S. Chattopadhyay, Right inverse $\Gamma$-semigroup, *Bull. Cal. Math. Soc.* **93**(2001), 435-442.

[2] S. Chattopadhyay, Right orthodox $\Gamma$-semigroup, *SEA Bull. Math.* **29**(2005), 23-30.

[3] R. Chinram, On quasi-gamma-ideals in gamma-semigroups, *ScienceAsia* **32**(2006), 351-353.

[4] R. Chinram, A note on quasi-ideals in Γ-semirings, *Int. Math. Forum*, **3**(2008), 1253-1259.

[5] R. Chinram and C. Jirojkul, On bi-Γ-ideals in Γ-semigroups, *Songklanakarin J. Sci. Technol.* **29**(2007), 231-234.

[6] R. Chinram and Y. Kemprasit, The intersection property of quasi-ideals in generalized rings of strictly upper triangular matrices, *East-West J. Math.* **3**(2001), 201-210.

[7] R. Chinram and C. Rungsaripipat, The intersection property of quasi-ideals in generalized rings of strictly upper triangular matrices over $m\mathbb{Z}$, *Advances in Algebra and Analysis* **1**(2006), 123-131.

[8] R. Chinram and P. Siammai, On green's relations for Γ-semigroups and reductive Γ-semigroups, *Int. J. Alg.* **2**(2008), 187-195.

[9] Y. Kemprasit and R. Chinram, Generalized rings of linear transformation having the intersection property of quasi-ideals, *Vietnam J. Math.* **30**(2002), 283-288.

[10] M. K. Sen, On Γ-semigroups, *Proceeding of International Symposium on Algebra and Its Applications*, Decker Publication, New York, (1981), 301-308.

[11] M. K. Sen and N. K. Saha, On Γ-semigroup I, *Bull. Cal. Math. Soc.* **78**(1986), 181-186.

[12] M. Shabir, A. Ali and S. Batool, A note on quasi-ideals in semirings, *SEA Bull. Math.* **27**(2004), 923-928.

[13] O. Steinfeld, On ideals quotients and prime ideals, *Acta. Math. Acad. Sci. Hung.* **4** (1953), 289-298.

[14] O. Steinfeld, Über die quasiideale von halbgruppen, *Publ. Math. Debrecen* **4**(1956), 262-275.

[15] O. Steinfeld, *Quasi-ideals in rings and semigroups*, Akadémiai Kiadó Budapest, 1978.


**Received: May 3, 2008**