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Random Convex Bodies in a Lattice of Parallelograms

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Abstract

We solve a problem of Buffon-type for an arbitrary "test body" K and a lattice of lines whose elementary tile $\mathcal{C}_0 = \cup_{i=1}^{m} \cup_{j=1}^{t} \mathcal{P}_i^{(j)}$, where $\mathcal{P}_i^{(j)}$ is a parallelogram of sides a_j and b_i and acute angle $\alpha \in]0, \pi/2]$.

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1 Introduction

In 1733, at a meeting of the Académie des Sciences de Paris, Buffon posed a problem that later on should become known as the famous "*Buffon needle problem*": in a room, the floor of which is merely divided by parallel lines, at a distance a apart, a needle of length $l < a$ is allowed to fall at random: which is the probability that the needle intersects one of the lines?

The solution, determined by Buffon by means of empirical methods, was $p = 2l/\pi a$. The problem and its solution were published in 1777, in the "*Comptes rendus de l'Académie des sciences de Paris*".

In 1812, Laplace extended the problem by considering a room paved with equal tiles, shaped as rectangles of sides a and b, with $l < min(a, b)$. The solution was $p = \frac{2l(a+b)-l^2}{\pi ab}$, and it is obvious that the probability of Buffon can be obtained from that of Laplace by letting $b \to +\infty$.

We restate now these problems in a slightly different form, which will be useful for several different extensions.

Let us denote by $\mathbf{E}_2$ be the Euclidean plane. By a lattice $\mathcal{R}$ in $\mathbf{E}_2$ we understand a sequence of closed and connected sets $\{\mathcal{C}_n\}_{n\in N}$ such that

1. $\bigcup_{n\in N} \mathcal{C}_n = \mathbf{E}_2$,
2. $Int(\mathcal{C}_i) \cap Int(\mathcal{C}_j) = \emptyset$, $\forall i, j \in N$ and $i \neq j$,
3. $\mathcal{C}_n = \gamma_n(\mathcal{C}_0)$, $\forall n \in N$, where γ_n are the elements of a discrete subgroup of the group of motions in $\mathbf{E}_2$ that leaves invariant the lattice.

The domain $\mathcal{C}_0$ is usually called the *fundamental tile* (or cell) of $\mathcal{R}$. Let us denote by K be a convex body (which means here a compact convex set) which we shall call *test body.* A general problem of Buffon type can be stated as follows:

"*Which is the probability $p_{K,\mathcal{R}}$ that the random convex body K, or more precisely, the random congruent copy of K, meets some of the boundary points of at least one of the domains $\mathcal{C}_n$?* "

In [1] A.Aleman, M.Stoka and T.Zamfirescu in considered Buffon's problem for an arbitrary convex test body K and certain lattices. When K is tangent to an oriented line g, then S_g will denote the orthogonal projection of S on g, and if φ is the angle between a given direction d related to the body and $\overline{SS_g}$, we set $p(\varphi) := |\overline{SS_g}|$, the distance from S to g.

The 2π-periodic extension function $p : R \to R$ will be called the *support function* with respect to the pair (K, d). We denote by L the function $L : R \to R$ given by $L(\varphi) := p(\varphi) + p(\varphi + \pi)$. We call L the *width* of the pair (K, d) in the direction φ. By construction L is a π-periodic function.

The goal of this paper is to compute the probability $p_{K,\mathcal{R}}$ that a convex body K, dropped at random, intersects a lattice $\mathcal{R}$ of lines whose elementary tile is a set of parallelograms $\mathcal{P}_i$ with sides a and b_i, for $i = 1, ..., m$ and acute angle $\alpha \in]0, \pi/2]$ as in the following picture, under the assumption that the support function is known. For other similar results see [4].

We denote by $\mathcal{M}$ the set of all convex test bodies congruent to K and with barycenter S within $\mathcal{C}_0$. We also assume that these convex test bodies are uniformly distributed, i.e. that the coordinates of S are a bidimensional random variable with uniform distribution in $\mathcal{C}_0$, and that the random variable φ is uniformly distributed in $[0, 2\pi]$, S and φ stochastically independent.

Finally we denote by $\mathcal{N}$ the set of convex bodies K, of diameter $Diam(K)$, which are completely contained in $\mathcal{C}_0$. As is well known then we can write the probability that the test body K intersects the boundary of one of the tiles of the lattice $\mathcal{R}$:

$$p_K = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1}$$

where μ is the Lebesgue measure.

The measures $\mu(\mathcal{N})$ and $\mu(\mathcal{M})$ can be computed using the elementary Kinematic measure in $\mathbf{E}_2$ [[9],p.126]

$$dK = dx \wedge dy \wedge d\varphi, \tag{2}$$

where x and y are the coordinates of $P \in K$ and φ is an angle of rotation.

2 Main Results

Now consider for fixed $\varphi \in [0, \pi]$ the set of points $P \in \mathcal{C}_0$ for with the body K with centroid P does not intersect the boundary $\partial\mathcal{C}_0$ and let $\mathcal{C}(\varphi)$ the topological closure of this open subset of $\mathcal{C}_0$. In the sequel we will assume that the body K is *small*[1] with respect to the lattice $\mathcal{R}$, using some restriction on the diameter $Diam(K)$ of K:

$$Diam(K) < min\Big\{a_1, ..., a_t, b_1, ..., b_m\Big\} \sin\alpha.$$

Theorem 2.1 *The probability that a convex body K of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\mathcal{R}$ is*

$$p_K = \frac{1}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}\Bigg\{\mathcal{L}\Bigg(m\sum_{i=1}^{m} b_i + t\sum_{i=1}^{m} b_i\Bigg) + \tag{3}$$

$$-\frac{mt}{\sin\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi\Bigg\}.$$

Proof. Let us consider the fundamental cell $\mathcal{C}$ of the lattice $\mathcal{R}$. We denote by $\mathcal{N}_i^{(j)}$ the set of all "test bodies" K whose barycentres are inside in the parallelogram $\mathcal{P}_i^{(j)}$.

Thus

[1]We say that the body K is small with respect to $\mathcal{R}$, if the polygons sides of $\mathcal{C}(\varphi)$ and $\mathcal{C}_0$ are pairwise parallel.

$$p_K = 1 - \frac{\sum_{j=1}^{t}\sum_{i=1}^{m}\mu(\mathcal{N}_i^{(j)})}{\mu(\mathcal{M})}, \tag{4}$$

where

$$\mu(\mathcal{M}) = 2\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha.$$

Let $\mathcal{R}_{\varphi}^{(i,j)}$ be the rectangle with sides parallel to those of $\mathcal{P}_i^{(j)}$, of lengths

$$a_j - L(\varphi)/\sin\alpha, \quad b_i - L(\varphi)/\sin\alpha.$$

We can write

$$\mu(\mathcal{N}_i^{(j)}) = \int_0^{2\pi} d\varphi \int\int_{(x,y)\in\mathcal{R}_{\varphi}^{(i,j)}} dxdy = \int_0^{2\pi} \frac{[a_j - L(\varphi)][b_i - L(\varphi)]}{\sin\alpha} d\varphi$$

By the fact that L is a π-periodic function and by Cauchy formula, i.e.

$$\int_0^{2\pi} L(\varphi)d\varphi = 2\mathcal{L},$$

we get

$$\sum_{j=1}^{t}\sum_{i=1}^{m}\mu(\mathcal{N}_i^{(j)}) = 2\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha - 2\mathcal{L}\left(m\sum_{j=1}^{t} a_j + t\sum_{i=1}^{m} b_i\right) +$$

$$+\frac{2mt}{\sin\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi.$$

When we replace the expression $\sum_{j=1}^{t}\sum_{i=1}^{m}\mu(\mathcal{N}_i^{(j)})$ in [4] we have the probability (3).

Corollary 2.2 *If S is a segment of constant length, the probability that S intersects one of the lines of the lattice $\mathcal{R}$ is*

$$p_K = \frac{1}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}\left\{2l\left(\sum_{j=1}^{t} a_j + \sum_{i=1}^{m} b_i\right) + \right. \tag{5}$$

$$\left. -mtl^2\Big[1 + \Big(\frac{\pi}{2} - \alpha\Big)\cot\alpha\Big]\right\}.$$

If E is an ellipse of half-axes ξ and ζ, the width function is given by

$$L(\varphi) = 2\sqrt{\xi^2 \sin^2 \varphi + \zeta^2 \cos^2 \varphi}.$$

Hence formula (3) gives the following

Corollary 2.3 *The probability that a random ellipse E of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\mathcal{R}$ is*

$$p_E = \frac{1}{\pi\Big(\sum_{j=1}^t a_j\Big)\Big(\sum_{i=1}^m b_i\Big)\sin\alpha}\Bigg\{\mathcal{L}\Bigg(m\sum_{j=1}^t a_j + t\sum_{i=1}^m b_i\Bigg) + \tag{6}$$

$$-\frac{4mt}{\sin\alpha}\int_0^\pi \sqrt{(\xi^2\sin^2\varphi + \zeta^2\cos^2\varphi)(\xi^2\sin^2(\varphi+\alpha) + \zeta^2\cos^2(\varphi+\alpha))}d\varphi\Bigg\}$$

The expression (6) extends the formula proved in [11] and formula 1.1 in [7].

In general for a convex body of constant width k, using Cauchy relation, we have

$$\mathcal{L} = \frac{1}{2}\int_0^{2\pi} k d\varphi = \pi k.$$

By this fact we obtain

Corollary 2.4 *If K has constant width k, the probability that K intersects one of the lines of the lattice $\mathcal{R}$ is*

$$p_{K,\mathcal{R}} = \Big(m\sum_{j=1}^t a_j + t\sum_{i=1}^m b_i\Big)\frac{k}{\Big(\sum_{j=1}^t a_j\Big)\Big(\sum_{i=1}^m b_i\Big)\sin\alpha} + \tag{7}$$

$$-\frac{mt}{\Big(\sum_{j=1}^t a_j\Big)\Big(\sum_{i=1}^m b_i\Big)\sin^2\alpha}k^2.$$

3 Dependence structure of the hitting events

We can look at the lattice $\mathcal{R}$ as the superposition of two elementary lattices of parallel lines: the lattice $\mathcal{R}_a$ of lines parallel to the side BC of $\mathcal{P}_1$, with equidistance $a\sin\alpha$, and the lattice $\mathcal{R}_b$ of lines parallel to the side DC of $\mathcal{P}_1$ and with equidistance $b_i \sin\alpha$. Hence

$$\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b.$$

We denote by E_a the event "a body test K intersects one of the lines of $\mathcal{R}_a$" and by E_b the event "a body test K intersects one of the lines of $\mathcal{R}_b$".

Theorem 3.1 *The probability $p^*_{K,\mathcal{R}}$ that a convex body K intersects at the same time two lines with different directions in the lattice $\mathcal{R}$ is*

$$p^*_{K,\mathcal{R}} = \frac{mt}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin^2\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi. \tag{8}$$

Proof. By simple remarks, using the same arguments in [2] and denoting by $p(E_a)$ and $p(E_b)$ the probabilities of the events E_a and E_b, we have

$$p(E_a) = \frac{t\mathcal{L}}{\pi\Big(\sum_{j=1}^{t} b_i\Big)\sin\alpha}, \quad p(E_b) = \frac{m\mathcal{L}}{\pi\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}.$$

But

$$p(E_a \cup E_b) = \frac{1}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}\Bigg\{\mathcal{L}\Bigg(m\sum_{j=1}^{t} a_j + t\sum_{i=1}^{m} b_i\Bigg)+ \\ -\frac{mt}{\sin\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi\Bigg\}.$$

Hence the probability $p(E_a \cap E_b)$ that K meets at the same time some line in $\mathcal{R}_a$ and some line in $\mathcal{R}_b$ is

$$p(E_a \cap E_b) = p(E_a) + p(E_b) - p(E_a \cup E_b).$$

Substituting the previous formulas we get the assertion.

Finally, imposing the condition of independence for the events E_a and E_b we get

Theorem 3.2 *The events E_a and E_b are independent if and only if*

$$\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi = \frac{\mathcal{L}^2}{\pi mt}. \tag{9}$$

Hence immediately we have

Corollary 3.3 *If Σ is a circle of constant radius δ with*

$$\delta < \frac{\sin\alpha}{2} min\{a_1, ..., a_t, b_1, ..., b_m\},$$

the events E_a and E_b are independent if and only if $(m, t) = (4, 4)$.

Finally

Corollary 3.4 *If K has constant width k, the events E_a and E_b are independent if and only the fundamental cell of the lattice $\mathcal{R}$ is given by a single parallelogram of sides a and b and acute angle $\alpha \in]0, \pi/2]$.*

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