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Backward (r, s)-Difference Operator r, s and Solving Difference Equations

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Abstract

In this note we introduce a new operator that we call it backward (r, s)-difference operator $\nabla_{r,s}$, defined as follow

$$\nabla_{r,s}\ y_n = r\ y_n - s\ y_{n+1}.$$

Then, we investigate some properties of this new operator, we find a shift exponential formula and use it in solving of the nonhomogeneous difference equations with constant coefficients, may be written in the following form

$$(\prod_{j=1}^{m} \nabla_{r_j,s_j})y_n = f_n.$$

Keywords: backward difference operator ∇, Forward (r, s)-difference operator $\nabla_{r,s}$, Difference equation , Shift exponential formula, Particular solution

1 Introduction

1.1. In Numerical Analysis, we use some linear operators: shift exponential operator E, "$Ef_j = f_{j+1}$", forward difference operator Δ,"$\Delta f_j = f_{j+1} - f_j$" and backward difference∇, "$\nabla f_j = f_j - f_{j-1}$". These operators are used in some topics of Numerical Analysis, particularly in interpolation, quadratures, difference equations, and so forth. [1], [2], [4].

Since E, Δ and ∇ are linear operator, then every order of them, their inversion$E^{-1}, \Delta{-1}, \nabla^{-1}$ and every polynomial of them are linear too. We know that difference equation appears in numerical solving of $ODE, PDE, IE, IDE, ...$ for example in numerical solving of initial problem "$y'(x) = f(x, y(x)), y(x_0) =$

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y_0", we find a difference equation of order one, if $f(x, y)$ is of degree one with respect to y, then corresponding difference equation is linear, otherwise, corresponding difference equation is nonlinear.

In this paper we find the particular solution of the nonhomogeneous difference equations with constant coefficients. Difference equations difference equations are written in the following form.
Under the backward difference operator ∇.

$$P(\nabla)y_n = 0, \qquad (homogeneous) \qquad (1)$$

$$P(\nabla)y_n = f_n. \qquad (nonhomogeneous) \qquad (2)$$

whereas P is polynomial.

In solving linear difference equations and finding general solution, we use the following theorems. [1], [2], [4], [5].

Theorem1 (superposition principle) Suppose that $y_1, y_2, ..., y_m$ are the (fundamental) solutions of the homogeneous difference equation $P(\nabla)y_n = 0$, then any linear combinations of them is a solution for it too.

Theorem2 Suppose that the complex-valued function "$y_n = y_1 + i\ y_2$"be a solution of equation $P(\nabla)y_n = 0$, then functions "y_1, y_2"also are solutions for it.

Theorem3 Let $y_h$be a solution for $P(\nabla)y_n = 0$ and $y_p$be a particular solution for $P(\nabla)y_n = f_n$, then "$y_c = y_h + y_p$" is a solution for $P(\nabla)y_n = f_n$ too.

2 Solution of the difference equations

Let "$y_n = r^n$" be a solution for equation (1), we have

Let "$y_n = r^n$" be a solution for equation (1), we have

$$P(1 - \frac{1}{r}) = 0. \qquad (3)$$

Where (3) is called the corresponding characteristic equation to equation (1).

Remark 1 All roots of the characteristic equations may be distinct real values, either some of them equal or some of them are conjugate complex number.

(i) If $r_1, r_2, , r_k$be distinct real roots to the characteristic equations, then the functions "$r_1^n, r_2^n, ..., r_k^n$" will be solutions of the homogeneous equations, these functions are linearly independent [3].

These functions are said the fundamental solutions of the homogeneous equation.

(ii) If$r_1 = r_2 = ... = r_m = r$ be the repeated roots of the characteristic equation (3), then the fundamental solutions of the homogeneous equation are: "$r^n, n\ r^n, n^2 r^n, ..., n^{m-1} r^n$" that are linearly independent [3].

(iii) If $r_{1,2} = \alpha \pm i\beta$ be two conjugate complex roots, the fundamental solutions of the homogeneous equation are,
"$y_1 = (\alpha^2 + \beta^2)^{n/2} cosn\varphi, \quad y_2 = (\alpha^2 + \beta^2)^{n/2} sinn\varphi,$

where $\varphi = tan^{-1}(\beta/\alpha)$" [3].

Example1 Find the fundamental solutions of the homogeneous difference equation:

$$(2\nabla^4 - 2\nabla^3 + \nabla^2) y_n = 0.$$

Solution It's characteristic equation is "$(1 - \frac{1}{r})^2 (2(1 - \frac{1}{r})^2 - 2(1 - \frac{1}{r}) + 1) = 0$". This polynomial equation has one double root "$r = l$" and two complex conjugate roots "$r = 1 \pm i$" therefore fundamental solutions may be written as follow:

$$y_1 = 1,\ y_2 = n,\ y_3 = 2^{\frac{n}{2}} cos\frac{n\pi}{4},\ y_4 = 2^{\frac{n}{2}} sin\frac{n\pi}{4}$$

Example2 Find the fundamental solutions of the following homogeneous difference equation

$$(21\nabla^2 - 10\nabla + 1) y_n = 0$$

Solution We have "$2(1 - \frac{1}{r})^2 - 101 - \frac{1}{r}) + 1 = 0$" which yields, "$r_1 = \frac{7}{6}, \quad r_2 = \frac{3}{2}$"and "$y_1 = (\frac{7}{6})^n, \quad y_2 = (\frac{3}{2})^n$"

Example3 Solve the following difference equation and find the fundamental solutions

$$(\nabla^3 - 11\nabla - 150)\ y_n = 0.$$

Solution The roots of the corresponding characteristic equation are "$r_1 = -\frac{1}{5}, r_{2,3} = \frac{1}{8}(1 \pm i)$" in the result, the fundamental solutions will be written as follow

$$y_1 = (-\frac{1}{5})^n, \quad y_2 = 2^{-\frac{5}{2}n} cos\frac{n\pi}{4}, \quad y_3 = 2^{-\frac{n}{2}n} sin\frac{n\pi}{4}$$

Example4 Evaluate the fundamental solutions of the following DE

$$"(1-2\nabla)^2(2-3\nabla)^2(2-\nabla)\ y_n = 0".$$

Solution We have $"(r-2)^2(r-3)^2+(r+1)=0"$which yields $"r_1 = r_2 = 2,\quad r_3 = -1,\quad r_4 = r_5 = 3,\quad r_6 = 0"$so the fundamental solutions may be written as follow $"y_1 = 2^n,\quad y_2 = n\ 2^n,\quad y_3 = (-1)^n,\quad y_4 = 3^n,\quad y_5 = n\ 3^n"$.

Lemma 1 Prove the accuracy of the following equalities.

$$\nabla \sum_{j=0}^{n} f_j = f_n \tag{4}$$

$$\frac{1}{\nabla}\ f_n = \sum_{j=0}^{n} f_j \tag{5}$$

Proof: It is easy, consider $\nabla \sum_{j=0}^{n} f_j = \sum_{j=0}^{n} f_j - \sum_{j=0}^{n-1} f_j = f_n$.
Identity (5) is the inversion of (4).

Remark 2 Each of above identities are used for finding particular solution of the nonhomogeneous difference equations with constant coefficients therefore we can solve each of the following equations

$$\nabla y_n = f_n, \quad \nabla^m y_n = f_n$$

Example 5 Find the particular solution of the following difference equation

$$\nabla y_n = n^2$$

Solution We can write

$$y_p = \frac{1}{\nabla} n^2 = \sum_{j=0}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$$

Example 6 Find the particular solution of the following difference equation

$$\nabla y_n = n^3 - n^2 + 2n + 2$$

Solution we can write

$$y_p = \frac{1}{\nabla} n^3 + 2n^2 + 3n + 1 = \sum_{j=1}^{n} (j^3 + 2j^2 + 3j + 1) = \frac{1}{12} n(3n^3 + 14n^2 + 43n + 32)$$

Example 7 Find the particular solution of the following difference equation

$$\nabla^3 y_n = 120n + 60$$

Solution By division operation we can write

$$y_p = \frac{1}{\nabla^2}\,(\frac{1}{\nabla}(120n+60)) = \frac{1}{\nabla^2}\sum_{j=1}^{n}(120j+60))\frac{1}{\nabla}(\frac{1}{\nabla}(60(n^2+2n)))$$

$$= \frac{1}{\nabla}(\sum_{j=1}^{n}(60(j^2+2j) = \frac{1}{\nabla}(10)2n^3+9n^2+7n))$$

$$= 10\sum_{j=1}^{n}(2j^3+9j^2+7j) = 5n(n+1)(n+3)(n+4)$$

DefinitionWe define the forward (r,s)-difference operator $\nabla_{r,s}$ as follow

$$\nabla_{r,s}y_n = ry_n - sy_{n-1} = (r - sE^{-1})y_n.$$

where y_n is the approximate value function $y(x)$ at point $x_n \in [x_0, x_m]$, then two operators "$\nabla_{r,s}$" and "$r - sE^{-1}$" are equivalent.

Corollary 2 $\nabla_{r,s}$is a linear operator and $\nabla_{1,1} \equiv 1 - E^{-1} \equiv \nabla$ and $\nabla_{1,1} \equiv \nabla \equiv 1 - E^{-1}$.

Example 9

$$\nabla_{r,s}((\frac{s}{r})^n y_n = r(\frac{s}{r})^n y_n - s(\frac{s}{r})^{n-1}y_{n-1} = \frac{s^n}{r^{n-1}}(y_n - y_{n-1} = r(\frac{s}{r})^n\nabla y_n$$

Example 10

$$\nabla_{1,5}(5^n(n^2-6n+10)) = 5^n(n^2-6n+10)-5.5^{n-1}((n-1)^2-6(n-1)+10 = 5^n(2n-1)$$

Four principal operations in vector space of operator $\nabla_{r,s}$

we define

(i) $\quad\nabla_{r_1,s} + \nabla_{r_2,s} \equiv \nabla_{r_1+r_2,s}$

(ii) $\quad\nabla_{r,s_1} + \nabla_{r,s_2} \equiv \Delta_{r,s_1+s_2}$

$(iii)\quad \nabla_{r_1,s} - \nabla_{r_2,s} \equiv \nabla_{r_1-r_2,s}$

$(iv)\quad \nabla_{r,s_1} - \nabla a_{r,s_2} \equiv \nabla_{r,s_1-s_2}$

$(v)\quad \nabla_{r_1,s_1} \times \nabla_{r_2,s_2} \equiv \nabla_{r_2,s_2} \times \nabla_{r_1,s_1}$

$(vi)\quad \frac{\nabla_{r_1,s_1}}{\nabla_{r_2,s_2}} \equiv \nabla_{r_1,s_1}(\frac{1}{\nabla_{r_2,s_2}}) \equiv (\frac{1}{\nabla_{r_2,s_2}})\nabla_{r_1,s_1}$

Particular type 1 Suppose that $s = r$, then
$\nabla_{r,s}\nabla_{r,s} \equiv \nabla^2_{r,s}, \ldots, \nabla_{r,s}(\nabla^m_{r,s}) \equiv \nabla^{m+1}_{r,s}$

In the sequel we define order and inversion for Backward (r, s)-difference operator.

$$\nabla^{-1}_{r,s} \equiv \frac{1}{\nabla_{r,s}} \quad s.t. \quad \frac{1}{\nabla_{r,s}} f_n = g_n \Leftrightarrow \nabla_{r,s}\ g_n = f_n$$

Remark 3 Addition operation and multiplication operation are commutative and associative, namely

$$(\nabla_{r_1,s_1} + \nabla_{r_2,s_2}) + \nabla_{r_3,s_3} \equiv \nabla_{r_1,s_1} + (\nabla_{r_2,s_2} + \nabla_{r_3,s_3}) \equiv \nabla_{r_1+r_2+r_3,s_1+s_2+s_3},$$

$$\nabla_{r_1,s_1} \times (\nabla_{r_2,s_2} \times \nabla_{r_3,s_3}) \equiv (\nabla_{r_1,s_1} \times \nabla_{r_2,s_2}) \times \nabla_{r_3,s_3}$$

Theorem 4 The backward (r, s)-difference operator is linear operator, in addition to, every order of it and every polynomial of$\nabla_{r,s}$ and inversion $\nabla^{-1}_{r,s}$ are linear too.

Proof: It is easy and left to the readers.

Lemma 2 Prove that accuracy each of the following identities:

$$\nabla_{r,s}(\sum_{j=0}^{n-1}((\frac{s}{r})^{n-j} f_j) = r f_n \tag{6}$$

$$\frac{1}{\nabla_{r,s}} f_n = \frac{1}{r}\sum_{j=1}^{n}((\frac{s}{r})^{n-j} f_j) \tag{7}$$

Proof: It is easy, consider

$$\nabla_{r,s}(\sum_{j=1}^{n}((\frac{s}{r})^{n-j} f_j) = r\sum_{j=1}^{n}((\frac{s}{r})^{n-j} f_j - s\sum_{j=0}^{n-1}((\frac{s}{r})^{n-1-j} f_j = r f_n$$

Identity (7) is the inversion of (6). Identities (6) and (7) are used in solution of NDE.

Remark 4 Under the backward (r,s)-difference operator $\nabla_{r,s}$, nonhomogeneous difference equation with constant coefficients is written in the following form

$$(\prod_{j=1}^{m} \nabla_{r_j,s_j})y_n = f_n \tag{8}$$

Whereas "$r_j, j = 1, 2, ..., m$" can be real distinct, repeated or complex numbers.

2.2. Many results by effective backward (r,s)-difference operator

(*i*) $\nabla_{r,s}(a^n) = (ra-s)a^{n-1} \Rightarrow \frac{1}{\nabla_{r,s}} a^n = \frac{a^{n+1}}{ra-s}$ $(a \neq \frac{s}{r})$.

In general

(*ii*) $\nabla^k_{r,s} a^n (ra-s)^{n-k} \Rightarrow \frac{1}{\nabla^k_{r,s}}(a^n) = \frac{a^{n+k}}{(ra-s)^k}$

Suppose that $P(ra-s) \neq 0$, then

(*iii*) $P(\nabla_{r,s})a^n = a^n P(\frac{ra-s}{a})$, $\frac{1}{P(\nabla_{r,s})} a^n = \frac{1}{P(\frac{ra-s}{a})} a^n$

Where P is a polynomial

Example 11 $\frac{1}{\nabla^4_{2,3}}(5^n) = \frac{5^{n+4}}{(2\times 5-3)^4} = \frac{5^{n+4}}{7^4}$

Lemma 3 Prove the accuracy of the following equalities

(*iv*) $\nabla_{r,s}(a^n y_n) = a^n \nabla_{ar,s} y_n$

(*v*) $\nabla^k_{r,s}(a^n y_n) = a^n \nabla^k_{ar,s} y_n$

Proof: Is easy.

Lemma 4 . Prove that

$$\nabla^m_{r,s} = ((\frac{s}{r})^n y_n) = r^m (\frac{s}{r})^n \nabla^m y_n \tag{9}$$

$$\frac{1}{\nabla^m_{r,s}}((\frac{s}{r})^n y_n) = \frac{1}{r^m}(\frac{s}{r})^n \frac{1}{\nabla^m} y_n \tag{10}$$

Proof: Equality (9) will be proved by mathematical, in addition to, (10) is the inversion of (9).

Example12 $\frac{1}{\nabla_{2,3}}((\frac{3}{2})^n n^2) = \frac{1}{2}(\frac{3}{2})^n \frac{1}{\nabla} n^2 = \frac{3^n}{2^{n+1}} \sum_{j=1}^{n} j^2 = \frac{3^n}{2^{n+1}} \times \frac{1}{6} n(n+1)(2n+1) = \frac{3^{n-1}}{2^{n+2}} n(n+1)(2n+1)$.

Particular case Suppose that $y_n = n^k$, then

$$\nabla_{r,s}^k (n^k (\frac{s}{r})^n = k! s^k (\frac{s}{r})^n \tag{11}$$

$$\frac{1}{\nabla_{r,s}^k}((\frac{s}{r})^n = \frac{n^k}{k! s^k}(\frac{s}{r})^n \tag{12}$$

Example 13 Evaluate $\frac{1}{\Delta_{2,3}^2}(\frac{3}{2})^n = \frac{1}{2!2^2} n^2 (\frac{3}{2})^n = \frac{1}{8} n^2 (\frac{3}{2})^n$

Theorem 5 (shift exponential) Let P be a polynomial, then

$$P(\nabla_{r,s})((\frac{s}{r})^n y_n) = (\frac{s}{r})^n P(r\nabla) y_n \tag{13}$$

$$\frac{1}{P\nabla_{r,s}}((\frac{s}{r})^n y_n) = (\frac{s}{r})^n \frac{1}{P(s\nabla)} y_n \tag{14}$$

Proof: The proof is easy by using Lemma 2.

Example 14 Find the particular solution of the following $D.E.$

$$(6E^2 - 13E + 6) y_n = 5(\frac{2}{3})^n$$

Solution This equation may be written as follow

$$(2E-3)(3E-2) y_n = \nabla_{2,3} \nabla_{3,2} y_n = 5(\frac{2}{3})^n$$

so

$$y_n = \frac{1}{\nabla_{2,3}}(\frac{1}{\nabla_{3,2}}(5(\frac{2}{3})^n)) = \frac{1}{\nabla_{2,3}}(\frac{5}{3} n (\frac{2}{3})^n = -\frac{3}{2}(\frac{2}{3})^n$$

Solution of NDE with constant coefficients

We know that every nonhomogeneous difference equation with order$m$ can be written in the form (8).Therefore each of the following forms may be written in the form of (8).

$$P(\nabla) y_n = f_n.$$

(8) is written as follows

$$y_p = \frac{1}{\prod_{j=1}^m \nabla_{r_j, s_j}} f_n = \frac{1}{\nabla_{r_m, s_m}}(\frac{1}{\Delta_{r_{m-1}, s_{m-1}}}(\ldots \frac{1}{\Delta_{r_1, s_1}} f_n \ldots)) \tag{15}$$

Example15 Find the particular solution of the following NDE

$$\nabla^2\nabla_{1,2}\nabla_{1,3}y_n = n2^n$$

Solution Write

$$y_p = \frac{1}{\nabla_{1,2}\nabla_{1,3}}\nabla^2(n2^n) = \frac{1}{\nabla_{1,2}}(\frac{1}{\nabla_{1,3}}(\frac{1}{\nabla}(n-1)2^{n+1}))) = \frac{1}{\nabla_{1,2}}(\frac{1}{\nabla_{1,3}}(n-2)2^{n+2}).$$

Remark5 In using of the identity (15), we may use iterative divisions, in addition to, we can use the decomposition fraction, consider

$$\frac{1}{\nabla_{r_1,s_1}\nabla_{r_2,s_2}} \equiv \frac{A_1}{\nabla_{r_1,s_1}} + \frac{A_2}{\nabla_{r_2,s_2}}, \quad A_1 = \frac{s_1}{r_2s_1-r_1s_2}, \quad A_2 = \frac{s_2}{r_1s_2-r_2s_1}$$

$$\frac{1}{\nabla_{r_1,s_1}\nabla_{r_2,s_2}\nabla_{r_3,s_3}} \equiv \frac{A_1}{\nabla_{r_1,s_1}} + \frac{A_2}{\nabla_{r_2,s_2}} + \frac{A_3}{\nabla_{r_3,s_3}},$$

$$A_1 = \frac{s_1^2}{(r_2s_1-r_1s_2)(r_3s_1-r_1s_3)}, A_2 = \frac{s_2^2}{(r_1s_2-r_2s_1)(r_3s_2-r_2s_3)},$$

$$A_3 = \frac{s_3^2}{(r_1s_3-r_3s_1)(r_2s_1-r_1s_2)}$$

In general $\frac{1}{\prod_{j=1}^{m}\nabla_{r_j,s_j}} \equiv \sum_{j=1}^{m}\frac{A_j}{\nabla_{r_j,s_j}}, \quad A_j = \frac{s_j^{m-1}}{\prod_{j\neq i=1}^{m}}(r_js_i - r_is_j).$

Example 16 Find the particular solution of the following $N.D.E.$

$$\nabla_{1,2}\nabla_{1,4}y_n = n2^n$$

Solution

$$y_p = \frac{1}{\nabla_{1,2}\nabla_{1,4}}(n2^n) = (\frac{1}{\nabla_{1,4}}(2^{n+1}n) - \frac{1}{\nabla_{1,2}}(n2^n)$$

$$= -(n+2)2^{n+1} - 2^{n-1}(n^2+n) = -2^{n-1}(n^2+5n+8).$$

Example 17 Find the particular solution of the following $N.D.E.$

$$(36 - 36E^{-1} + 11E^{-2} - E^{-3})y_n = 6^{n-3}(6n^2 - 48n - 80)$$

Solution Write $(2 - E^{-1})(3 - E^{-1})(6 - E^{-1})y_n = 6^{n-3}(6n^2 - 48n - 80)$ and $y_p = \frac{1}{\nabla_{2,1}\nabla_{3,1}\nabla_{6,1}}(6^{n-3}(6n^2 - 48n - 80)) = 6^{n+1}n^3.$

Remark 6 If $r_1 = r_2 = ... = r_m = r$, then we can use only iterative divisions.

Remark 7 Let $P(\nabla_{r,s}y_n = a^nf_n$, then by change of variable "$y_n = Y_na^n$" and using the shift exponential formula, we obtain the following $N.D.E.$

$$P(\nabla_{r,\frac{s}{a}})Y_n = f_n \tag{16}$$

Remark 8 $P(\nabla_{r,s}y_n = a^n(a \neq \frac{s}{r})$,
If $P(a-\frac{s}{r}) \neq 0$, then

$$y_p = \frac{1}{P(a-\frac{s}{r})}a^n$$

If $P(a-\frac{s}{r}) = P'(a-\frac{s}{r}) = ... = P^{k-1}(a-\frac{s}{r}) = 0, P^(k)(a-\frac{s}{r}) \neq 0$, then

$$y_p = \frac{1}{P^(k)(a-\frac{s}{r})}n^k a^n$$

3 Concluding

The backward (r, s)-difference operator method is a new method in solution of nonhomogeneous difference equations with constant coefficients and we can use it in solving of the following equation

$$P(\nabla)y_n = f_n$$
$$P(\nabla) \equiv \sum_{j=1}^{m} a_j \nabla^j$$

where a_j's are constants .

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