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Quasi σ-Rigid Rings

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Abstract. In the present paper, we introduce the notion of quasi σ-rigid rings for an automorphism σ of a ring R. Some properties of a σ-ring R that has the form $R[x;\sigma,\delta]$ are obtained, for example if R is a quasi-Baer ring then $R[x;\sigma,\delta]$ is a quasi-Baer ring when R is a quasi σ-rigid ring.

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Throughout this paper, we assume that R is an associative ring with unity.

In [2], Krempa introduced the notion of the rigid ring. An endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. According to Hong-Kim-Kwak [1], R is said to be a *σ-rigid ring* if there exits a rigid endomorphism σ of R. Recall that a ring R is called a *reduced ring* if it has no nonzero nilpotent elements. It is well known that σ-rigid rings are reduced rings.

For an endomorphism σ of a ring R, the additive map $\delta : R \to R$ is called a *σ-derivation* if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

for any $a, b \in R$. The *Ore extension* $R[x; \sigma, \delta]$ of R is a ring which is obtained by giving the polynomial ring over R with multiplication

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$.

Let R be a ring. An ideal I of R is called a *σ-ideal* if $\sigma(I) \subseteq I$. Let σ be an automorphism of R. For a σ-ideal I of R, we call I *quasi σ-rigid ideal* of R if $aR\sigma(a) \subseteq I$ implies $a \in I$ for any $a \in R$. In the present paper, we will characterize quasi σ-rigid rings. Some properties of a σ-ring R with those of the ring $R[x; \sigma, \delta]$ are obtained.

For an endomorphism σ of a ring R and an ideal I of R, the following proposition is proved easily.

Proposition 1 *Suppose $\sigma(I) = I$. Then the following statements are equivalent:*
(1) *For all $a \in R$, if $aR\sigma(a) \subseteq I$ then $a \in I$.*
(2) *For all $a \in R$, if $\sigma(a)Ra \subseteq I$ then $a \in I$.*
(3) *For all $n \in \mathbb{Z}$ and $a \in R$, if $aR\sigma^n(a) \subseteq I$ then $a \in I$.*
(4) *For all $J \lhd R$, if $J\sigma(J) \subseteq I$ then $J \subseteq I$.*

Recall that, I *quasi σ-rigid ideal* of R if $\sigma(I) \subseteq I$ and $aR\sigma(a) \subseteq I$ implies $a \in I$ for any $a \in R$. It is easy to see that, the intersection of all quasi σ-rigid ideals is again a quasi σ-rigid ideal by Proposition 1. Thus a ring with an automorphism σ contains the smallest quasi σ-rigid ideal I such that R/I is semisimple.

Theorem 2 *Let R be a ring and I be a σ- ideal of R. Then the following statements are equivalent:*
(1) *I is a quasi σ-rigid ideal of R.*
(2) *For any $a + I \in R/I$, if $(a + I)R/I\bar{\sigma}(a + I) \subseteq I$ then $a \in I$, where $\bar{\sigma} : R/I \to R/I$, defined by $a + I \to \bar{\sigma}(a + I) = \sigma(a) + I$ for $a \in R$, is an automorphism of R/I.*

Proof. Clear. □

Lemma 3 *Let R be a ring, σ be an automorphism of R and I be a quasi σ-rigid ideal of R. Then $Ker(\sigma) \subseteq I$, $\sigma^{-1}(I) = I$, i.e., σ-induces an injective endomorphism of R/I and I is a semiprime ideal of R.*

Proof. Let σ be an automorphism of R and I be a quasi σ-rigid ideal of R. Then $Ker(\sigma) \subseteq I$ and $I \subseteq \sigma^{-1}(I)$. Now, let $x \in \sigma^{-1}(I)$. Then $\sigma(a) \in I$. This follows that $aR\sigma(a) \in I$, i.e., $a \in I$.

Let $yRy \subseteq I$ for $y \in R$. Then we have $yr\sigma(y)R\sigma(yr\sigma(y)) \subseteq I$ for any $r \in R$. This implies that $yr\sigma(y) \in I$, i.e., $y \in I$. □

Proposition 4 *Let R be a ring, σ be an automorphism of R and I be a σ-rigid ideal of R. Then the following statements are equivalent:*
(1) *I is a quasi σ-rigid ideal of R.*
(2) *For any $x \in R$, $\sigma(x)Rx \subseteq I$ implies $x \in I$.*

Proof. (1) $\Rightarrow$ (2) By Lemma 3.
(2) $\Rightarrow$ (1) Assume that $\sigma(x)Rx$ implies $x \in I$ for any $x \in R$. Let $xRx \subseteq I$. Then, for any $r \in R$, we have $\sigma(\sigma(x)rx)R\sigma(x)rx = \sigma^2(x)\sigma(r)\sigma(xRx)rx$. This implies that $\sigma(x)rx \in I$ and so $x \in I$. Now we can obtain that I is a semiprime ideal. Hence, if $xR\sigma(x) \subseteq I$ then $\sigma(x)Rx \subseteq I$ and so $x \in I$. That is I is a quasi σ-rigid ideal of R. □

The following theorem generalizes to Proposition 4.

Theorem 5 *Let R be a ring, σ be an automorphism of R and I be a σ- ideal of R. Then the following statements are equivalent:*
(1) *I is a quasi σ-rigid ideal of R.*
(2) *For any ideal J of R, $J\sigma(J) \subseteq I$ implies $J \subseteq I$.*

Proof. Similar to Lemma 3. □

Let R be a ring and I be a quasi σ-rigid ideal of R. In Lemma 3, we proved that I is a semiprime ideal of R. Now we give one of the main results of this study.

Theorem 6 *Let R be a ring, σ be an automorphism of R and I be a σ-rigid ideal of R. If I is a σ- ideal of R then $I[x; \sigma, \delta]$ is a semiprime ideal of $R[x; \sigma, \delta]$.*

To prove, we need the following lemma.

Lemma 7 *Let R be a ring σ be an automorphism of R, and I be a quasi σ-rigid ideal of R.*
(1) *If $xRy \subseteq I$ then $xR\sigma^n(y) \subseteq I$ and $\sigma^n(x)Ry \subseteq I$for every positive integer*

n and $x, y \in R$.
(2) If $xR\sigma^m(y) \subseteq I$ then $xRy \subseteq I$ for some positive integer m and $x, y \in R$.
(3) Assume that I is a σ-rigid ideal of R and $0 \neq p(x) = a_0+a_1x+\cdots+a_nx^n$ and $0 \neq q(x) = b_0+b_1x+\cdots+a_mx^m \in R[x;\sigma,\delta]$ then $p(x)R[x;\sigma,\delta]q(x) \subseteq I[x;\sigma,\delta]$ if and only if $a_iRb_j \subseteq I$ for all $0 \leq i \leq n$ and $0 \leq j \leq n$.

Proof. (1) Let $xRy \subseteq I$ for any $x, y \in R$. For any $r \in R$, $yr\sigma(x)R\sigma(br\sigma(x)) = yr\sigma(x)\sigma(R)\sigma(y)\sigma(r\sigma(x)) = yr\sigma(xRy)\sigma(r\sigma(x)) \subseteq I$. This implies that $yR\sigma(x) \subseteq I$ since I is a quasi σ-rigid ideal of R. By Lemma 3, I is a semiprime ideal, then $\sigma(x)Ry \subseteq I$ and so $\sigma(x)Ry \subseteq I$.
(2) Assume that $xR\sigma^m(y) \subseteq I$ for some positive integer m and $x, y \in R$. Then, by (1), $\sigma^m(xRy) = \sigma^m(x)\sigma^m(R)\sigma^m(y) = \sigma^m(x)R\sigma^m(y) \subseteq I$. By Lemma 3, $\sigma^{m-1}(xRy) \subseteq \sigma^{-1}(I) = I$. Then $xRy \subseteq I$ by continuing this step.
(3) By (2). □

Proof of Theorem 6 For $p(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\sigma,\delta]$, we assume that $p(x)R[x;\sigma,\delta]p(x) \subseteq I[x;\sigma,\delta]$. By Lemma 7, we have $a_iRa_i \subseteq I$ for all $0 \leq i \leq n$. By Lemma 3, I is a semiprime ideal, we have $a_i \in I$ for all $0 \leq i \leq n$. This implies that $p(x) \in I[x;\sigma,\delta]$.

A ring R is said to be *σ-semiprime* if whenever A is an ideal of R and m is an integer such that $A\sigma^n(A) = 0$ for all $n \leq n$ then $A = 0$ (see [3]).

Theorem 8 *Let R be a ring. If R is a quasi σ-rigid ring then $R[x;\sigma,\delta]$ is a semiprime ring.*

Note that the converse of Theorem 8 is not true.

Example 9 Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and let $\sigma : R \to R$ defined by $\sigma(x, y) = (y, x)$. Note that the only proper ideal σ-ideal $I = \{(0, 0)\}$. Then I is semiprime ideal. Since $(1, 0)R\sigma(1, 0) \subseteq I$ but $(1, 0) \notin I$, I is not a quasi σ-rigid ideal. On the other hand $R[x;\sigma]$ is not a semiprime ring.

Lemma 10 *Let R be a quasi σ-rigid ring. $0 \neq p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $0 \neq q(x) = b_0 + b_1x + \cdots + a_mx^m \in R[x;\sigma,\delta]$. Then $p(x)R[x;\sigma,\delta]q(x) = 0$ if and only if $a_iRb_j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq n$.*

Proof. By Lemma 7. □

Recall that a ring is called *quasi-Baer* if the right annihilator of each right ideal of R is generated by an ideal.

Theorem 11 *Let R be a quasi σ-rigid ring. If R is a quasi-Baer ring then $R[x;\sigma,\delta]$ is a quasi-Baer ring.*

Proof. Let I be an ideal of $R[x;\sigma,\delta]$ and C_I denote the set of all coefficients of elements of I. Then C_I is an ideal of R. Since R is a quasi-Baer ring, then $ann_R(C_I) = eR$ for some $e^e = e \in R$ and e is central. Then $C_I Re = 0$. By Lemma 10, for any $p(x =\in I$, $p(x)R[x;\sigma,\delta]e = 0$. Then $Ie = 0$ and $e \in ann_{R[x;\sigma,\delta]}(I)$. This implies that $eR[x;\sigma,\delta] \subseteq ann_{R[x;\sigma,\delta]}(I)$. Let $0 \neq q(x) = b_0 + b_1x + \cdots + b_mx^m \in ann_{R[x;\sigma,\delta]}(I)$. Then $p(x)R[x;\sigma,\delta]q(x) = 0$. By Lemma 10, we have $b_0, b_1, ..., b_mx^m \in ann_R(C_I) = eR$. This implies that $ann_{R[x;\sigma,\delta]}(I) = eR[x;\sigma,\delta]$. □

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