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A New Analytical Method for Finding General and Exact Solution of Nonlinear Kaup–Kupershmidt Equation

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Abstract

In this article, a new but powerful analytical method i.e. He's Exp–Function method is introduced to solve stiff nonlinear problem. He's Exp–Function method is employed to compute an exact solution of nonlinear differential equation governing the problem. Nonlinear Kaup–Kupershmidt (KK) equation is used as an example to illustrate the simple solution procedures. It has been attempted to show the capabilities and wide-range applications of the Exp–Function method.

Keywords: Exp–Function method; Kaup–Kupershmidt equation, nonlinear equation, Exact solution, Partial differential equation

1. Introduction

Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Most scientific problems and phenomena such as heat transfer occur nonlinearly. Except in a limited number of these problems, we have difficulty in finding their exact analytical solutions. Therefore, approximate analytical solutions were introduced.

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However, in recent years, analytical solution [1] has considerably been developed and is used for nonlinear partial equations such as Kaup–Kupershmidt (KK) equation [2] that have special kind of solution.

Recently Ji-Huan He [3-14] introduced some new methods such as variation iteration method (VIM), homotopy perturbation method (HPM) and Exp–Function method to solve these equations [18-22]. Exp–Function method is very strong for solving stiff nonlinear equations. Other authors such as Zhu SD. [15-16] and Zhang s [17] were working in this field.

In this Letter, we propose to present implementation of Exp–Function method to Kaup–Kupershmidt (KK) equation. Having exact solution of the special form of the corresponding equations [2] would provide us to have an admissible comparison of the results, which supports the applicability, accuracy, and efficiency of the proposed method.

The Kaup–Kupershmidt (KK) equation is in the form of [2]:

$$u_t + 45u^2u_x - 15pu_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0, \tag{1}$$

2. Basic idea of Exp-function method

We first consider nonlinear equation in the form of:

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \dots) = 0, \tag{2}$$

Introducing a complex variation defined as

$$\eta = kx + \omega t \quad , u = U(\eta), \tag{3}$$

and therefore, equation (1) is changed to ODE in the form of

$$N(U, -k\omega U', kU', k^2U'', k^2\omega^2U'', -k^2\omega U'', \dots) = 0. \tag{4}$$

and then, solution of $U(\eta)$ is in the form of

$$U(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \dots + a_{-d}\exp(-d\eta)}{a_p \exp(p\eta) + \dots + a_{-q}\exp(-q\eta)} \tag{5}$$

where c, d, p and q are positive integers which are unknown to be determined later, a_n and b_n are unknown constants.

3. Application of Exp–Function method

Introducing a complex variation η defined as equation (3), equation (1) becomes an ordinary differential equation in the form of

$$\omega U' + 45kU^2U' - 15pk^3U'U'' - 15k^3UU''' + k^5U''''' = 0, \tag{6}$$

In order to determine values of *c* and *p*, we balance the linear term of highest order U''''' with the highest order nonlinear term U^2U' in equation (6), we have

$$U''''' = \frac{c_1 \exp[(31p + c)\eta] + ...}{c_2 \exp[32 p\eta] + ...}, \tag{7}$$

$$U^2 U' = \frac{c_3 \exp[(3c + p)\eta] + ...}{c_4 \exp[4 p\eta] + ...} \times \frac{\exp[28 p\eta]}{\exp[28 p\eta]} = \frac{c_3 \exp[(3c + 29 p)\eta] + ...}{c_4 \exp[32 p\eta] + ...} \tag{8}$$

where c_i are coefficients introduced for simplicity. Balancing highest order of Exp–function in equations (7) and (8), we have

$$31p + c = 29p + 3c \tag{9}$$

which leads to the result

$$p = c \tag{10}$$

Similarly, to determine values of *d* and *q*, we balance the linear term of lowest order in equation (6).

$$U''''' = \frac{... + d_1 \exp[-(31q + d)\eta]}{... + d_2 \exp[-32 q\eta]} \tag{11}$$

and

$$U^2 U' = \frac{... + d_3 \exp[-(3d + q)\eta]}{... + d_4 \exp[-4q\eta]} \times \frac{\exp[-28q\eta]}{\exp[-28q\eta]} = \frac{... + d_3 \exp[-(3d + 29q)\eta]}{... + d_4 \exp[-32q\eta]} \tag{12}$$

where d_i are determined coefficients introduced only for simplicity. Balancing lowest order of Exp–function in equations (11) and (12), we have

$$-(3d + 29q) = -(31q + d) \tag{13}$$

This leads to the result

$$q = d \tag{14}$$

3.1. Case2: $p = c = 2$, $d = q = 2$

For simplicity, we set $p = c = 1$ and $d = q = 1$. then equation (5) leads to

$$U(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{15}$$

Substituting equation (15) into equation (6), and by the help of MAPLE, we have

$$\begin{aligned} &\frac{1}{A}\{C_5 e^{5\eta} + C_4 e^{4\eta} + C_3 e^{3\eta} + C_2 e^{2\eta} + C_1 e^{\eta} + C_0 + C_{-1} e^{-\eta} \\ &+ C_{-2} e^{-2\eta} + C_{-3} e^{-3\eta} + C_{-4} e^{-4\eta} + C_{-5} e^{-5\eta}\} = 0, \end{aligned} \tag{16}$$

where we have

$$A = (\exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6 \tag{17}$$

and C_n are coefficients of $\exp(n\eta)$. Equating the coefficients of $\exp(n\eta)$ to zero, we have

$$\begin{aligned} &C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0, C_5 = 0, \\ &C_0 = 0, C_{-1} = 0, C_{-2} = 0, C_{-3} = 0, C_{-4} = 0, C_{-5} = 0, \end{aligned} \tag{18}$$

Solving the system of equation (18) simultaneously yields

$$b_0 = b_0,\ b_{-1} = \frac{1}{4}b_0^2, \qquad \omega = -\frac{k^5\left(5p^2+10p+5p\sqrt{p^2+4p-4-12}\right)}{8},$$

$$a_1 = \frac{\left(p+2+\sqrt{p^2+4p-4}\right)k^2}{12},\ a_{-1} = \frac{1}{48}\left(p+2+\sqrt{p^2+4p-4}-12\right)k^2 b_0^2, \tag{19}$$

$$a_0 = \frac{5}{6}\frac{k^2 b_0\left(p-2+\sqrt{p^2+4p-4}\right)}{p-\sqrt{p^2+4p-4}},$$

Substituting these results into (15), we obtain the following generalized exact solution of equation (6).

$$U(\eta) = \frac{\left(\frac{p+\delta+2}{12}\right)k^2\exp(\eta) + \frac{5}{3}\left(\frac{p+\delta-2}{2(p+\delta)}\right)k^2 b_0 + \left(\frac{p+\delta+2}{48}\right)k^2 b_0^2\exp(-\eta)}{\exp(\eta)+b_0+\frac{1}{4}b_0^2\exp(-\eta)}, \tag{20}$$

$$U(\eta) = \left(\frac{p+2+\sqrt{p^2+4p-4}}{12}\right)k^2 + \frac{4k^2 b_0\left(p-2+\sqrt{p^2+4p-4}\right)}{(4\exp(\eta)+4b_0+b_0^2\exp(-\eta))(p-\sqrt{p^2+4p-4})} \tag{21}$$

where, b_0 is an arbitrary constant parameter and $\delta = \sqrt{p^2+4p-4}$. In addition, when k and ω are imaginary numbers, the obtained exact solution can be converted into periodic solution, we write

$$k = iK,\ \omega = i\Omega \tag{22}$$

By using the transformation

$$\exp[\ (kx+\omega t)] = \cos[\ Kx+\Omega t] + i\sin[\ Kx+\Omega t] \tag{23}$$

and

$$\exp[\ -(kx+\omega t)] = \cos[\ Kx+\Omega t] - i\sin[\ Kx+\Omega t] \tag{24}$$

Then Equation (22) becomes

$$u(x,t) = \left(\frac{p+2+\delta}{12}\right)K^2 + \frac{4K^2 b_0\left(p-2+\delta\right)}{\cos[Kx+\Omega t](4+(p-\delta)\,b_0^2)+4b_0+i\sin[Kx+\Omega t](4-(p-\delta)b_0^2)}, \tag{25}$$

If we search for a periodic solution, the imaginary part in equation (25) must be zero, which requires that

$$(4-(p-\delta)b_0^2) = 0 \tag{26}$$

and

$$b_0 = \pm\sqrt{\frac{4}{p-\delta}}, \tag{27}$$

Substituting equation (27) into (25) we can write

$$u(x,t)=\left(\frac{p+2+\delta}{12}\right)K^2+\frac{K^2\sqrt{\frac{4}{p-\delta}}(p-2+\delta)}{2\cos[Kx+\Omega t]+4\sqrt{\frac{4}{p-\delta}}},\tag{28}$$

or

$$u(x,t)=\left(\frac{p+2+\delta}{12}\right)K^2-\frac{K^2\sqrt{\frac{4}{p-\delta}}(p-2+\delta)}{2\cos[Kx+\Omega t]+4\sqrt{\frac{4}{p-\delta}}},\tag{29}$$

3.2. Case2: $p=c=2$**,** $d=q=2$

As mentioned above the values of c and d can be freely chosen, we set $p=c=2$ and $d=q=2$, then the trial function, equation (5) becomes

$$U(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)}{b_2\exp(2\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)+b_{-2}\exp(-2\eta)}\tag{30}$$

There are some free parameters in equation (30), we set $b_2=1$, $b_1=0$ and $b_{-1}=0$ for simplicity, the trial function, equation (30) is simplified as follows

$$U(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)}{\exp(2\eta)+b_0+b_{-2}\exp(-2\eta)}\tag{31}$$

By the same manipulation as illustrated above, we obtain

$$\begin{gathered}\omega=-2k^5(5p^2+10p+5p\sqrt{p^2+4p-4}-12),\ a_2=\frac{(p+2+\sqrt{p^2+4p-4})k^2}{3}\\ a_0=\frac{10}{3}\frac{k^2b_0(p-2+\sqrt{p^2+4p-4})}{p-\sqrt{p^2+4p-4}},\ a_{-2}=\frac{1}{12}(p+2+\sqrt{p^2+4p-4})k^2b_0^2,\\ b_{-2}=\frac{1}{4}b_0^2,\quad a_{-1}=0,\quad a_1=0,\ b_0=b_0,\end{gathered}\tag{32}$$

Substituting Equation (32) into (31) yields the following exact solution

$$\begin{aligned}U(\eta)=&\left(\left(\frac{p+2+\sqrt{p^2+4p-4}}{3}\right)k^2\exp(2\eta)+\frac{10}{3}\frac{k^2b_0(p-2+\sqrt{p^2+4p-4})}{p-\sqrt{p^2+4p-4}}\right.\\ &\left.+\frac{1}{4}\left(\frac{p+2+\sqrt{p^2+4p-4}}{3}\right)k^2b_0^2\exp(-2\eta)\right)\Big/\left(\exp(2\eta)+b_0+\frac{1}{4}b_0^2\exp(-2\eta)\right),\end{aligned}\tag{33}$$

or

$$u(x,t)=\left(\frac{p+2+\sqrt{p^2+4p-4}}{3}\right)k^2 + \frac{16k^2 b_0(p-2+\sqrt{p^2+4p-4})}{\left(4\exp(2(kx+\omega t))+4b_0+b_0^2\exp(-2(kx+\omega t))\right)(p-\sqrt{p^2+4p-4})}, \tag{34}$$

where k and ω are imaginary numbers

$$u(x,t)=\left(\frac{p+2+\delta}{3}\right)K^2 + \frac{16K^2 b_0\left(p-2+\delta\right)}{4\cos[2\,(Kx+\Omega\,t)](4+(p-\delta)\,b_0^2)+4b_0+i\sin[2(Kx+\Omega t)](4-(p-\delta)b_0^2)}, \tag{35}$$

If we search for a periodic solution, the imaginary part in equation (35) must be zero, which requires that

$$b_0=\pm\sqrt{\frac{4}{p-\delta}} \tag{36}$$

and

$$\delta=\sqrt{p^2+4p-4}, \tag{37}$$

Substituting Equation (37) into (35), we have the following new periodic solution of equation (1):

$$u(x,t)=\left(\frac{p+2+\delta}{3}\right)K^2+\frac{K^2\left(p-2+\delta\right)\sqrt{\frac{4}{p-\delta}}}{4\cos^2[(Kx+\Omega\,t)]+(\sqrt{\frac{4}{p-\delta}}-2)}, \tag{38}$$

and

$$u(x,t)=\left(\frac{p+2+\delta}{3}\right)K^2-\frac{K^2\left(p-2+\delta\right)\sqrt{\frac{4}{p-\delta}}}{4\cos^2[(Kx+\Omega\,t)]-(\sqrt{\frac{4}{p-\delta}}+2)}, \tag{39}$$

3.2. Case3: $p=c=2$**,** $d=q=1$

As mentioned above the values of c and d can be freely chosen, we set $p=c=2$ and $d=q=1$, then the trial function, equation (5) becomes

$$U(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{b_2\exp(2\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{40}$$

There is free parameter in equation (40). We set $b_2=1$. For simplicity, trial function of equation (40) is simplified as follows

$$U(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(2\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{41}$$

By the same manipulation as illustrated above, we obtain

$$a_1=\frac{5}{6}\frac{k^2b_1(p-2+\sqrt{p^2+4p-4})}{p-\sqrt{p^2+4p-4}},\ a_0=\frac{1}{48}(p+2+\sqrt{p^2+4p-4})k^2b_1^2,$$
$$a_2=\frac{(p+2+\sqrt{p^2+4p-4})k^2}{12},\quad b_{-1}=0,\ b_0=\frac{1}{4}b_1^2,\ a_{-1}=0,\ b_1=b_1, \tag{42}$$
$$\omega=-\frac{k^5(5p^2+10p+5p\sqrt{p^2+4p-4}-12)}{8},$$

Substituting equation (42) into (41) yields the following exact solution

$$U(\eta)=\left(\left(\frac{p+2+\sqrt{p^2+4p-4}}{12}\right)k^2\exp(2\eta)+\frac{10}{3}\frac{k^2b_1(p-2+\sqrt{p^2+4p-4})}{p-\sqrt{p^2+4p-4}}\exp(\eta)\right.$$
$$\left.+\frac{1}{4}\left(\frac{p+2+\sqrt{p^2+4p-4}}{12}\right)k^2b_1^2\right)/\left(\exp(2\eta)+b_1\exp(\eta)+\frac{1}{4}b_1^2\right), \tag{43}$$

or

$$u(x,t)=\left(\frac{p+2+\sqrt{p^2+4p-4}}{12}\right)k^2$$
$$+\frac{4k^2b_1(p-2+\sqrt{p^2+4p-4})}{\left(4\exp(2(kx+\omega t))+4b_1\exp(kx+\omega t)+b_1^2\right)(p-\sqrt{p^2+4p-4})}, \tag{44}$$

Substituting $b_1=b_0$ in equations (43) and (44), we can recover the equations (20) and (21).

4. Conclusion

In this Letter we solved Nonlinear Kaup–Kupershmidt (KK) equation by the Exp–function method. It can be concluded that the Exp–Function method is very powerful and efficient technique in finding exact solutions for wide classes of problems. The result shows that the Exp–Function method is a powerful new method for nonlinear equations having many applications in engineering. It shows that the Exp–function method is a very efficient method. We sincerely hope this method can be applied in a wider range. In this work, we used well-known software Maple to calculate the series and the rational functions obtained from the proposed techniques.

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