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# On the Jacobsthal Numbers by Matrix Methods

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**Abstract**

In this study, we define the Jacobsthal $F$-matrix and Jacobsthal $M$-matrix similar to Fibonacci Q-matrix. After, using this matrix representation we have found the some equalities and the Binet-like formula for the Jacobsthal numbers.

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**Keywords:** Fibonacci numbers, Jacobsthal numbers, Matrix method

## 1 Introduction

The Fibonacci, the Lucas numbers and their generalization have many interesting properties and applications to almost every fields of science and art. For the prettiness and rich applications of these numbers and their relatives one can see science and the nature [1, 5] and [7-17].

In [12], Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. In so doing, he shows that if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

then

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix},$$

where $u_n$ represents the $n$thFibonacci number.

In 1960, Charles H. King studied on the following $Q$-matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

in his Ms thesis. He showed that

$$\det(Q) = -1$$

and

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Moreover, it is easy seen that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \text{ (Cassini's formula).}$$

Above equalities demonstrate that there is a very close link between the matrices and Fibonacci numbers. More generally, there are some relations between the integer sequences and matrices [10, 15].

In [10], some properties obtained related Fibonacci and Lucas numbers by matrix method such that $m, n \geq 1$ as follows:

*i)* $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$,
*ii)* $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$,
*iii)* $F_{m+n} = F_mF_{n+1} + F_{m-1}F_n$,
*iv)* $F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$,
*v)* $F_{m+1}L_n + F_mL_{n-1} = L_{m+n}$,
*vi)* $2F_{m+n} = F_mL_n + F_nL_m$,
*vii)* $2L_{m+n} = L_mL_n + 5F_mF_n$

where $F_n$ and $L_n$ denote the $n$th Fibonacci and Lucas numbers, respectively. Furthermore, in [10] generalized characteristic equation and Binet formula for the Fibonacci and Lucas numbers using matrix method are given by

$$x^2 - L_nx + (-1)^n = 0,$$
$$x = (L_n \pm \sqrt{5}F_n)/2$$

and

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2}, \ \beta^n = \frac{L_n - \sqrt{5}F_n}{2}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

We know that the Fibonacci sequence is also a Lucas sequence $U_n(a, b)$ where $a$ and $b$ are equal to $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ respectively. That is, we notice that $a = \alpha$ and $b = \beta$ for Fibonacci and Lucas numbers [3, 4] , [16] and [18].

Microcontrollers (and other computers) use conditional instructions to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This winds up being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits, ..., which are exactly the Jacobsthal numbers [19].

In [6], the Jacobsthal and the Jacobsthal-Lucas sequences $J_n$ and $j_n$ are defined by the recurrence relations

$$J_0 = 0,\ J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

and

$$j_0 = 2,\ j_1 = 1,\ j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2$$

respectively.

In additation, the Jacobsthal numbers are the numbers obtained from the $U_n$ in the Lucas sequence with $a = 2$ and $b = -1$. Similarly, the Jacobsthal-Lucas numbers are the numbers obtained by the $V_n$ in the the Lucas sequence with $a = 2$ and $b = -1$ [4, 18].

The first eleven terms of the Jacobsthal sequence $\{J_n\}$ are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171 and 341. This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1}$$

The first eleven terms of the sequence $\{j_n\}$ are 2, 1, 5, 7, 17, 31, 65, 127, 257, 511 and 1025. This sequence is given by the formula

$$j_n = 2^n + (-1)^n \text{ [2]}. \tag{2}$$

In [6] Cassini-like formulas of Jacobsthal and Jacobsthal-Lucas numbers are given by

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}, \tag{3}$$
$$j_{n+1}j_{n-1} - j_n^2 = 3^2 \cdot (-1)^{n+1} \cdot 2^{n-1}. \tag{4}$$

In this study, we have defined Jacobsthal $F$-matrix by

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

It is easy to seen that, it can be write

$$\begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} = F \begin{bmatrix} J_n \\ J_{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix} = F \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}$$

where $J_n$ and $j_n$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively.

Furthermore, we have defined Jacobsthal $M$-matrix by

$$M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}. \tag{6}$$

Throughout this paper, $J_n$ and $j_n$ denote the $n$th Jacobsthal and Jacobsthal-Lucas numbers.

## 2 The Matrix Representation of Jacobsthal Numbers

In this section, we present two different matrix representation of Jacobsthal numbers which is called Jacobsthal $F$-matrix and Jacobsthal $M$-matrix. Using this representations we obtain the determinants and elements of $F^n$ and $M^n$, and we get Cassini-like formula for the Jacobsthal numbers. After, we calculated the generalized characteristic roots and Binet formula of the matrix $F^n$. Finally, we get sum formulas for the Jacobsthal numbers using these matrices.

**Theorem 1** *Let $F$ be a matrix as in (5). Then*

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}. \tag{7}$$

**Proof.** We will use the principle of mathematical induction (PMI). When $n = 1$,

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} J_2 & 2J_1 \\ J_1 & 2J_0 \end{bmatrix}$$

so the result is true. We assume it is true for any positive integer $n = k$:

$$F^k = \begin{bmatrix} J_{k+1} & 2J_k \\ J_k & 2J_{k-1} \end{bmatrix}.$$

Now, we show that it is true for $n = k + 1$. Then, we can write

$$\begin{aligned} F^{k+1} &= F^k F = \begin{bmatrix} J_{k+1} & 2J_k \\ J_k & 2J_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} J_{k+2} & 2J_{k+1} \\ J_{k+1} & 2J_k \end{bmatrix} \end{aligned}$$

and the result follows. ■

**Corollary 2** *For all positive integers n, following equalities hold:*
*i)* $\det(F^n) = (-2)^n$,
*ii)* $J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}$ *(Cassini-like formula).*

**Proof.** It is easy to see that

$$\det(F) = -2.$$

Then, it can be write

$$\begin{aligned} \det(F^n) &= \det(F) \cdot \det(F) \cdot \cdots \cdot \det(F) \\ &= (-2)^n. \end{aligned}$$

The determinant of $F^n$ in (7) and from (i), (ii) follows. ■

**Theorem 3** *Let n be an integer. The well-known Binet-like formula of the Jacobsthal numbers is*

$$J_n = \frac{2^n - (-1)^n}{3}.$$

**Proof.** Let the matrix $F$ be as in (5). If we calculate the eigenvalues and eigenvectors of the matrix $F$ are

$$\lambda_1 = 2, \ \lambda_2 = -1$$

and

$$v_1 = (2, \ 1), \ v_2 = (-1, \ 1),$$

respectively. We notice that eigenvalues of the matrix $F$ are seen

$$\lambda_1 = a, \ \lambda_2 = b$$

as in [18]. Then, we can diagonalize of the matrix $F$ by

$$D = P^{-1}FP$$

where

$$P = (v_1^T, v_2^T) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \text{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

From the properties of the similar matrices, we can write

$$D^n = P^{-1}F^nP$$

where $n$ is any integer. Furthermore, we can write

$$F^n = PD^nP^{-1} \tag{8}$$

where

$$D^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}.$$

By (7) and (8), we get

$$\begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{2\cdot 2^n+(-1)^n}{3} & \frac{2\cdot 2^n-2\cdot(-1)^n}{3} \\ \frac{2^n-(-1)^n}{3} & \frac{2^n+2\cdot(-1)^n}{3} \end{bmatrix}.$$

Thus, the proof is completed. ■

Consequently, limiting ratio of the successive Jacobsthal number is

$$\begin{aligned} a &= \lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \lim_{n\to\infty} \frac{((2^{n+1} - (-1)^{n+1})/3}{(2^n - (-1)^n)/3} \\ &= 2. \end{aligned}$$

**Theorem 4** *The generalized characteristic roots of $F^n$ are*

$$\lambda_{1,2} = (j_n \pm 3J_n)/2$$

*where $\lambda_1$ and $\lambda_2$ denote the characteristic roots of $F^n$. Then,*

$$J_n = \frac{2^n - (-1)^n}{3} \quad \textit{and} \quad j_n = 2^n + (-1)^n.$$

**Proof.** If we write the characteristic polynomial of $F^n$, we get

$$\begin{aligned} \det(F^n - \lambda I) &= \lambda^2 - (J_{n+1} + 2J_{n-1})\lambda + 2(J_{n+1}J_{n-1} - J_n^2) \\ &= \lambda^2 - j_n \cdot \lambda + (-2)^n, \end{aligned}$$

by identities $J_{n+1} + 2J_{n-1} = j_n$ and $J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}$.Thus, the characteristic equation of $F^n$ is

$$\lambda^2 - j_n \cdot \lambda + (-2)^n = 0 \tag{9}$$

and we get the generalized characteristic roots as following:

$$\lambda_{1,2} = \frac{j_n \pm \sqrt{j_n^2 - 4(-2)^n}}{2}.$$

Since $j_n^2 - 4(-2)^n = 9J_n^2$, it can be write $\lambda_{1,2} = (j_n \pm 3J_n)/2$. Consequently,

$$a^n = \frac{j_n + 3J_n}{2} \text{ and } b^n = \frac{j_n - 3J_n}{2}.$$

Thus, we give the Binet-like formula by matrix method for the Jacobsthal and Jacobsthal-Lucas numbers given in (1) and (2) by

$$J_n = \frac{2^n - (-1)^n}{3} \text{ and } j_n = 2^n + (-1)^n.$$

■

We know the matrix equation in (7), then we can write,

$$\frac{F^n}{J_{n-1}} = \begin{bmatrix} \frac{J_{n+1}}{J_{n-1}} & 2\frac{J_n}{J_{n-1}} \\ \frac{J_n}{J_{n-1}} & 2 \end{bmatrix}.$$

Since $\lim_{n\to\infty} (J_{n+1}/J_n) = a$, it follows that

$$\lim_{n\to\infty} \frac{F^n}{J_{n-1}} = \begin{bmatrix} a^2 & 2a \\ a & 2 \end{bmatrix} = \begin{bmatrix} a+2 & 2a \\ a & 2 \end{bmatrix}.$$

If we compute the determinant of both sides, we reach the characteristic equation of the Jacobsthal $F$-matrix as follows:

$$a^2 - a - 2 = 0.$$

**Theorem 5** *Let $M$ be $2 \times 2$ matrix as in (6), then*

$$M^n = \begin{bmatrix} J_{2n+1} & 2J_{2n} \\ J_{2n} & 2J_{2n-1} \end{bmatrix}$$

*for $n \geq 1$.*

**Proof.** It can be show easily by PMI. ■

**Corollary 6** *For all positive integers n, following equalities are valid:*
*i)* $\det(M^n) = 2^{2n}$,
*ii)* $J_{2n+1}J_{2n-1} - J_{2n}^2 = 2^{2n-1}$.

**Proof.** Proofs are easily seen similar to Corollary 2. ■

**Theorem 7** *For any integer $n \geq 1$,*

$$\sum_{i=0}^{n} J_i = \frac{1}{2}(J_{n+2} - 1).$$

**Proof.** When $n = 1$, equation (9) becomes $\lambda^2 - \lambda - 2 = 0$, which is the charracteristic equation of Jacobsthal $F$-matrix. Notice that $F^2 - F - 2I = 0$ (from the Cayley-Hamilton Theorem).

Now, we have following equation

$$(I + F + F^2 + ... + F^n)(F - I) = F^{n+1} - I. \tag{10}$$

Since $\det(F - I) = -2 \neq 0$, $F - I$ is invertible. Since $F^2 = F + 2I$ and $F^2 - F = 2I$, we can write $F(F - I) = 2I$. Thus, $\frac{1}{2}(F - I) = F^{-1}$. If we multiply both sides of equation (10) by $(F - I)^{-1} = \frac{1}{2}F$ we get

$$\begin{aligned} I + F + F^2 + ... + F^n &= \frac{1}{2}(F^{n+1} - I) \cdot F \\ &= \frac{1}{2}(F^{n+2} - F). \end{aligned}$$

Equating the (2,1) entry of both side, we have

$$\sum_{i=0}^{n} J_i = \frac{1}{2}(J_{n+2} - 1).$$

This completes the proof. ■

**Theorem 8** *Let $n$ and $k$ be a positive integer. Then, following relation between the Jacobsthal and Lacobsthal-Lucas numbers*

$$j_{n+k} = J_{k+1}j_n + 2J_kj_{n-1}$$

*is valid.*

**Proof.** From the definition of the Jacobsthal-Lucas numbers,

$$\begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix} = F \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}$$

can be written easily. If we multiply both side with $F^k$, we get

$$F^k \begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix} = F^{k+1} \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}.$$

Using (7) we obtain

$$\begin{bmatrix} j_{n+k+1} \\ j_{n+k} \end{bmatrix} = \begin{bmatrix} J_{k+2}j_n + 2J_{k+1}j_{n-1} \\ J_{k+1}j_n + 2J_kj_{n-1} \end{bmatrix}.$$

Thus, the proof is completed. ■

**Theorem 9** *Let $m$ and $n$ are positive integers. Then, following equalities are hold:*

*i)* $J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n$,
*ii)* $J_{2n} = J_n J_{n+1} + 2J_n J_{n-1} = J_n j_n$,
*iii)* $J_{2n+1} = J_{n+1}^2 + 2J_n^2$,
*iv)* $(-1)^n \cdot 2^{n-1} J_{m-n} = J_m J_{n-1} - J_{m-1}J_n$.

**Proof.** Let the matrix $F$ as in (5). Since $F^{m+n} = F^m F^n$, we can write

$$\begin{bmatrix} J_{m+n+1} & 2J_{m+n} \\ J_{m+n} & 2J_{m+n-1} \end{bmatrix} = \begin{bmatrix} J_{m+1}J_{n+1} + 2J_m J_n & 2(J_{m+1}J_n + 2J_m J_{n-1}) \\ J_m J_{n+1} + 2J_{m-1}J_n & 2(J_m J_n + 2J_{n-1}J_{m-1}) \end{bmatrix}.$$

Thus, equalities *(i)*, *(ii)* and *(iii)* are easily seen. If we calculate $F^{-n}$, we get

$$F^{-n} = \frac{1}{(-2)^n} \begin{bmatrix} 2J_{n-1} & -2J_n \\ -J_n & J_{n+1} \end{bmatrix}.$$

Since $F^{m-n} = F^m F^{-n}$, we can write

$$\begin{bmatrix} J_{m-n+1} & 2J_{m-n} \\ J_{m-n} & 2J_{m-n-1} \end{bmatrix} = \frac{(-1)^n}{2^n} \begin{bmatrix} 2(J_{m+1}J_{n-1} - J_m J_n) & -2(J_{m+1}J_n - J_m J_{n+1}) \\ 2(J_m J_{n-1} - J_{m-1}J_n) & -2(J_m J_n - J_{m-1}J_{n+1}) \end{bmatrix}$$

and $(iv)$ immediately seen. ■

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