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A Note on Frame Sequences in Hilbert Spaces

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Abstract

A frame for Hilbert space H has been decomposed into two infinite subsequences and observed that if one of the subsequence is a frame for H, then the other need not be a frame for H. A necessary and sufficient condition under which the other subsequence is a frame for H has been given. Also, it has been proved that if these two subsequences have finite excess then they are frame sequences for H. Finally, a sufficient condition for a frame of non-zero elements under which it is a bounded frame has been given.

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1 Introduction

Fourier transform has been a major tool in analysis for over a century. It has a lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was a localized time frequency representation which has this information encoded in it. In 1946, Dennis Gabor [10] filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952, Duffin and Schaeffer [8] introduced frames for Hilbert spaces to study some problems in non harmonic Fourier series. In fact, they abstracted the fundamental notion of Gabor for studying signal processing. The idea of Duffin and Schaeffer did not generate much interest outside non harmonic Fourier series. But, after the landmark paper of Daubechies, Grossmann and Meyer [7] in 1986, the theory of frames began to be more widely studied. Frames now a days are main tools for use in signal processing, image processing, data compression, sampling theory, optics, filter banks, signal detection, time frequency analysis, wavelet analysis, etc.

Casazza [4] and Benedetto and Fickus [2] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. For results on signal reconstruction without phase information, one may refer to [1]. Frame sequences are useful in cases where we are interested only in expansions in subspaces. For literature regarding frames and frame sequences, one may refer to [3, 5, 6, 9, 11].

In the present paper, a frame for Hilbert space H has been decomposed into two infinite subsequences and observed that if one of the subsequence is a frame for H, then the other need not be a frame for H. A necessary and sufficient condition under which the other subsequence is a frame for H has been given. Also, it has been proved that if these two subsequences have finite excess then they are frame sequences for H. Finally, a sufficient condition for a frame of non-zero elements under which it is a bounded frame has been given.

2 Preliminaries

Throughout the paper, H will denote an infinite dimensional Hilbert space, $\{n_k\}$ an infinite increasing sequence in $\mathbb{N}$, $[f_n]$ the closed linear span of $\{f_n\}$.

Definition 2.1. A sequence $\{f_n\} \subset H$ is called a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{2.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the frame $\{f_n\}$. The inequality (2.1) is called the *frame inequality*. The frame $\{f_n\} \subset H$ is called *tight* if it is possible to choose A, B satisfying inequality (2.1) with $A = B$ and is called *normalized tight* if $A = B = 1$. The frame $\{f_n\} \subset H$ is called *exact* if removal of an arbitrary f_n renders the collection $\{f_n\}$ no longer a frame for H. The frame $\{f_n\} \subset H$ is called *near exact* if it can be made exact by removing finitely many elements from it. Also a near exact frame is called proper if it is not exact.

A frame $\{f_n\} \subset H$ is called bounded frame if there exists a constant $\delta > 0$ such that $\|f_n\| > \delta$ for all n. A sequence $\{f_n\} \subset H$ is called a *frame sequence* if it is a frame for $[f_n]$. Excess of a sequence $\{f_n\}$ is defined as

$$e(\{f_n\}) = \sup\{|G| : G \subset \{f_n\} \text{ and } [f_n] = [\{f_n\} \setminus G]\}$$

where $|G|$ is cardinality of G.

3 Main Results

Let $\{f_n\}$ be a frame for H and $\{n_k\}$ be any infinite increasing sequence in $\mathbb{N}$. Then $\{f_{n_k}\}$ need not be a frame sequence. We give the following example in this regard.

Example 3.1. Let $\{f_n\}$ be the sequence of unit orthonormal vectors in H.

(a) For any infinite increasing sequence $\{n_k\}$ in $\mathbb{N}$, $\{f_{n_k}\}$ is a frame sequence.

(b) Define $\{g_n\}$ in H by $g_n = \dfrac{1}{\sqrt{n}} f_n, n \in \mathbb{N}$. Let $n_k = n_{k-1} + (k-1)$, $k \in \mathbb{N}$ and $n_0 = 1$. Then $\{n_k\}$ is an infinite increasing sequence in $\mathbb{N}$. Define a sequence $\{h_n\}$ in H by

$$h_1 = g_1, h_{n_k} = h_{n_k+1} = h_{n_k+2} = \ldots = h_{n_{k+1}-1} = g_k, \quad k \geq 2.$$

Then $\{h_n\}$ is a tight non-exact frame for H. But $\{h_{n_k}\} = \{g_k\}$ is not a frame sequence.

The following observations arise in wake of the Example 3.1.

Observations. **(I)** Let $\{f_n\}$ be a frame for H and $\{n_k\}$ be any infinite increasing sequence in $\mathbb{N}$. Then $\{f_{n_k}\}$ is a frame sequence if $\{f_n\}$ is exact.

(II) If $\{g_n\} \subset H$ is a sequence defined by $g_{2n-1} = f_n$, $g_{2n} = f_n$, for all $n \in \mathbb{N}$, where $\{f_n\}$ is a sequence of unit orthonormal vectors in H, then for every infinite increasing sequence $\{n_k\}$, $\{g_{n_k}\}$ is a frame sequence but $\{g_n\}$ is not exact.

(III) Let $\{f_n\}$ be a proper near exact frame for H and $\{n_k\}$ be any infinite increasing sequence in $\mathbb{N}$. Then $\{f_{n_k}\}$ is a near exact frame sequence for H. Indeed, since $\{f_n\}$ is a proper near exact frame, there exists a non empty finite subset say M of $\mathbb{N}$ such that $\{f_n\}_{n \in \mathbb{N} \setminus M}$ is exact. Let $\{n_k\}$ be any infinite increasing sequence in $\mathbb{N}$. Then, by Observation (I), $\{f_{n_k}\}_{n_k \in \mathbb{N} \setminus M}$ is an exact frame sequence for H. Therefore, $\{f_{n_k}\}$ is a near exact frame sequence for H.

(IV) The condition of near exactness in Observation(III) is not necessary Observation(II).

(V) Let $\{f_n\}$ be a frame for H and let $\{n_k\}$ and $\{m_k\}$ be two infinite increasing sequences in $\mathbb{N}$ such that $\{n_k\} \cup \{m_k\} = \mathbb{N}$. If $\{f_{n_k}\}$ is a frame sequence for H, then $\{f_{m_k}\}$ need not be a frame sequence for H. Indeed, let $\{h_n\}$ be the sequence as given in Example 3.1 (b). Let $m_k = m_{k-1} + k$, for all $k \in \mathbb{N}$, where $m_1 = 2$. Then $\{m_k\}$ is an infinite increasing sequence in $\mathbb{N}$. Let $\{n_k\}$ be the infinite increasing sequence such that $\{n_k\} = \mathbb{N} \backslash \{m_k\}$. Then $\{h_n\}$ is a frame for H and $\{f_{n_k}\}$ is a frame sequence for H. But $\{f_{m_k}\}$ is not a frame sequence for H.

(VI) Let $\{f_n\}$ be a frame for H. If $f_n \notin [f_i]_{i \neq n}$, for all $n \in \mathbb{N}$, then $\{f_n - f_{n+1}\}$ is not a frame for H. Indeed, if $\{f_n - f_{n+1}\}$ is a frame for H, then for each $f \in H$,

$$f = \sum_{n=1}^{\infty} \alpha_n (f_n - f_{n+1}) = \sum_{n=1}^{\infty} \alpha_n f_n - \sum_{n=1}^{\infty} \alpha_n f_{n+1}$$

Since $\{f_n\}$ is a frame for H, $f = \sum\limits_{n=1}^{\infty} \beta_n f_n$. So

$$\sum_{n} \alpha_n f_{n+1} = \sum_{n=1}^{\infty} (\alpha_n - \beta_n) f_n.$$

Also, since $\{f_n\}$ is exact, $\alpha_1 \neq \beta_1$. Therefore

$$f_1 = \sum_{n=1}^{\infty} \left(\frac{\alpha_n - (\alpha_{n+1} - \beta_{n+1})}{\alpha_1 - \beta_1} \right) f_{n+1}$$

This is a contradiction.

(VII) The condition that $f_n \notin [f_i]_{i \neq n}$ in (VI) is sufficient but not necessary (Example 3.1(b)).

(VIII) In view of (VI), one may observe that if $\{f_n\}$ is a frame for H such that $\{f_n + f_{n+1}\}$ is also a frame for H. Then $\{f_n\}$ is not near exact frame for H. Indeed, if $\{f_n\}$ is near exact, for some finite set S of $\mathbb{N}$, then $\{f_n\}_{n \in \mathbb{N} \backslash S}$ is exact. Note that $\{f_n + f_{n+1}\}$ have finitely many more elements than $\{f_n + f_{n+1}\}_{n \in \mathbb{N} \backslash S}$. Therefore, $\{f_n + f_{n+1}\}_{n \in \mathbb{N} \backslash S}$ and $\{f_n + f_{n+1}\}$ would be same in nature. Considering $\{f_n\}_{n \in \mathbb{N} / S}$ as an exact frame of H, we can therefore conclude that $\{f_n + f_{n+1}\}_{n \in \mathbb{N} \backslash S}$ is not a frame for H, i.e., $\{f_n + f_{n+1}\}$ is not a frame for H. This is a contradiction.

(IX) Let $\{f_n\}$ be a frame for H with bounds A and B, let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences with $\{m_k\} \cup \{n_k\} = \mathbb{N}$ such that $[f_{m_k}] = [f_{n_k}]$. If $\{f_{n_k}\}$ is a frame sequence for H with bounds

A' and B' such that $A' < A$, then $\{f_{m_k}\}$ is also a frame sequence for H. Indeed, if A' and B' be frame bounds for $\{f_{n_k}\}$. Then

$$A'\|f\|^2 \leq \sum_k |\langle f, f_{n_k}\rangle|^2 \leq B'\|f\|^2, \text{ for all } f \in [f_{n_k}].$$

Therefore, by the frame inequality for the frame $\{f_n\}$,

$$(A - A')\|f\|^2 \leq \sum_n |\langle f, f_{m_k}\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in [f_{m_k}].$$

In view of Observation(IX), we have the following result.

Theorem 3.2. *Let $\{f_n\}$ be any frame for H and let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences with $\{m_k\} \cup \{n_k\} = \mathbb{N}$. Also let $\{f_{m_k}\}$ be frame for H. Then $\{f_{n_k}\}$ is a frame for H if and only if there exists a bounded linear operator $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ such that*

$$T\{\langle f_{n_k}, f\rangle\} = \{\langle f_{m_k}, f\rangle\}, \quad f \in H$$

Proof. Let A be a lower bound of the frame $\{f_{n_k}\}$. Now since

$$\begin{aligned} \sum |\langle f_{m_k}, f\rangle|^2 &= \|T\{\langle f_{n_k}, f\rangle\}\| \\ &\leq \|T\| \sum |\langle f_{n_k}, f\rangle|^2 \end{aligned}$$

we have

$$\sum |\langle f_{n_k}, f\rangle|^2 \geq \frac{\sum |\langle f_{m_k}, f\rangle|^2}{\|T\|} \geq \frac{A}{\|T\|}\|f\|^2 .$$

Hence $\{f_{n_k}\}$ is also a frame for H.

Conversely, let $\{f_{n_k}\}$ be a frame for H. Then there exist operators

$$T_1 : \ell_2(\mathbb{N}) \to H \quad \text{given by} \quad T_1\{\langle f_{n_k}, f\rangle\} \to f \quad \text{and}$$

$$T_1^* : H \to \ell_2(\mathbb{N}) \quad \text{given by} \quad T_1^* f \to \{\langle f_{n_k}, f\rangle\}.$$

Also, since $\{f_{m_k}\}$ is a frame for H, there exist operators

$$T_2 : \ell_2(\mathbb{N}) \to H \quad \text{given by} \quad T_2\{\langle f_{m_k}, f\rangle\} \to f \quad \text{and}$$

$$T_2^* : H \to \ell_2(\mathbb{N}) \quad \text{given by} \quad T_2^* f \to \{\langle f_{m_k}, f\rangle\}.$$

Then $T = T_2^* T_1 : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ is a bounded linear operator such that

$$T\{\langle f_{n_k}, f\rangle\} = \{\langle f_{m_k}, f\rangle\}, \quad f \in H .$$

Theorem 3.3. *Let $\{f_n\}$ be any frame for H and let $\{m_k\}$, $\{n_k\}$ be two infinite increasing sequences with $\{m_k\} \cup \{n_k\} = \mathbb{N}$. Let $H' = [f_{m_k}] \cap [f_{n_k}]$. If H' is a finite dimensional space, then $\{f_{m_k}\}$ and $\{f_{n_k}\}$ are frame sequences for H.*

Proof. Let $\{\ell_k\}$ be any finite subsequence of $\{n_k\}$ such that $H' = [f_{\ell_k}]$. Since H' is finite dimensional, $\{f_{\ell_k}\}$ is a frame for H'. Let A' and B' be the bounds of the frame $\{f_{\ell_k}\}$. Consider $\{f_{n_k}\}$. Let $f \in [f_{n_k}]$ be any element. Now, if $f \perp H'$, then

$$\sum |\langle f, f_n\rangle|^2 = \sum |\langle f, f_{n_k}\rangle|^2 \geq A\|f\|^2 .$$

Also, if $f \in H'$, then

$$\sum |\langle f, f_{n_k}\rangle|^2 \geq \sum |\langle f, f_{\ell_k}\rangle|^2 \geq A'\|f\|^2$$

Otherwise, we have

$$\begin{aligned} f &= \sum \alpha_k f_{n_k} \\ &= \sum \alpha_i f_i + \sum \alpha_j f_j, \quad i \in \{n_k\} \setminus \{\ell_k\}, j \in \{\ell_k\} \\ &= f' + f'', \quad \text{where } f' \perp H' \text{ and } f'' \in H'. \end{aligned}$$

Thus

$$\begin{aligned} \sum |\langle f, f_{n_k}\rangle|^2 &= \sum |\langle f, f_i\rangle|^2 + \sum |\langle f, f_j\rangle|^2, \quad i \in \{n_k\} \setminus \{\ell_k\}, j \in \{\ell_k\} \\ &= \sum |\langle f' + f'', f_i\rangle|^2 + \sum |\langle f' + f'', f_j\rangle|^2 \\ &= \sum |\langle f', f_i\rangle|^2 + \sum |\langle f'', f_j\rangle|^2 \\ &\geq A\|f'\|^2 + A'\|f''\|^2 \\ &\geq \min\left(\frac{A}{2}, \frac{A'}{2}\right) \|f\|^2 . \end{aligned}$$

Hence $\{f_{n_k}\}$ is a frame sequence. Similarly, we can show $\{f_{m_k}\}$ is also a frame sequence □

Remark 3.4. The condition in Theorem is not necessary (Observation (II)).

In the following result, we give a sufficient condition for a frame of non-zero elements in terms of frame sequences for its exactness.

Theorem 3.5. *Let $\{f_n\}$ be a frame for H with optimal bounds A and B such that $f_n \neq 0$, for all $n \in \mathbb{N}$. If for every infinite increasing sequence $\{n_k\}$ in $\mathbb{N}$, $\{f_{n_k}\}$ is a frame sequence with optimal bounds A and B, then $\{f_n\}$ is an exact frame.*

Proof. Suppose $\{f_n\}$ is not exact. Then there exists an $m \in \mathbb{N}$ such that $f_m \in [f_i]_{i \neq m}$. Let $\{n_k\}$ be an increasing sequence given by $n_k = k$, $k = 1, 2 \ldots, m-1$ and $n_k = k+1$, $k = m, m+1, \ldots$. Since $\{f_{n_k}\}$ is a frame for H with bounds A and B,

$$A\|f\|^2 \leq \sum_{n \neq m} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H .$$

Therefore, by frame inequality for the frame $\{f_n\}$,

$$|\langle f, f_m\rangle|^2 = 0, \quad \text{for all } f \in H.$$

In particular, $|\langle f_m, f_m\rangle|^2 = 0$. This given $f_m = 0$, a contradiction.

The condition in Theorem 3.5 is not necessary (Example 3.7).

Example 3.6. Let $\{f_n\}$ be the sequence of unit orthonormal vectors in H. Define $\{g_n\} \subset H$ by

$$g_1 = f_1; \quad g_2 = \frac{1}{2} f_2 \ \text{ and } \ g_k = f_k, \quad \text{for all } \ k \geq 3.$$

Then $\{g_n\}$ is an exact frame for H with bounds $\frac{1}{2}$ and 1. Let $n_1 = 1$ and $n_k = k+1$, for all $k \geq 2$. Then $\{n_k\}$ is an infinite increasing sequence such that $\{f_{n_k}\}$ is a frame sequence with both bounds equal to 1.

Now, we give sufficient condition for a frame of non-zero elements under which it is a bounded frame.

Theorem 3.7. *Let $\{f_i\}_{i\in I}$, $f_i \neq 0$, for all $i \in I$ (I an index set) be a frame for Hilbert space H. Let for any infinite set $J \subset I$, $\{f_j\}_{j\in J}$ be a frame sequence with bounds A_J and B_J respectively. If $\inf_J A_J > 0$, then $\{f_i\}_{i\in I}$ is a bounded frame for H.*

Proof. Suppose $\{f_i\}_{i\in I}$ is not a bounded frame. If $\|f_i\| > A$, for all $i \in I$, where $A = \inf_J A_J$, then $\{f_i\}_{i\in I}$ is a bounded frame for H. Let $k_1 \in I$ be such that $\|f_{k_1}\| < A$. Choose ϵ_1 such that $\|f_{k_1}\| < \epsilon_1 < A$. If $\|f_i\| \geq \|f_{k_1}\|$, for all $i \in I$, then $\{f_i\}_{i\in I}$ is a bounded frame for H. Let $k_2 \in I$ be such that $\|f_{k_2}\| < \|f_{k_1}\|$. Choose ϵ_2 such that $\|f_{k_2}\| < \epsilon_2 < \|f_{k_1}\|$.
Continuing this way, we get an decreasing sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $\epsilon_n < A$, for all $n \in \mathbb{N}$ and an infinite set of indices $K = \{k_1, k_2, \ldots, k_n, \ldots\}$ such that $\|f_{k_n}\| < \epsilon_n$, for all $n \in \mathbb{N}$.

Let $H_1 = [f_{k_n}]_{k_n\in K}$ and $H_0 = [f_{k_n}]^{\perp}$. Then $H = H_0 \oplus H_1$.

Case (i) $\dim H_1$ is infinite.

Since $H_1 = [f_{k_n}]_{k_n\in K}$, there exists an infinite set of indices $K' = \{k'_1, k'_2, \ldots, k'_n, \ldots\}$ such that $\{f_{k'_n}\}_{k'_n\in K'}$ are linearly independent. Therefore by given hypotheses, $\{f_{k'_n}\}_{k'_n\in K'}$ is a frame sequence with lower bound $A_{K'} \geq A$. So

$$A\|f\|^2 \leq \sum_{k_{n'}\in K'} |\langle f, f_{k'_n}\rangle|^2, \quad \text{for all } f \in [f_{k'_n}]_{k_{n'}\in K'}.$$

Note that $\|f_{k'_i}\| < A$ for any $i \in \mathbb{N}$. Then, by Proposition 4.6 in [3], $f_{k'_i} \in [f_{k'_n}]_{\substack{k'_n\in K' \\ k'_n\neq k'_i}}$ which is not possible as $\{f_{k'_n}\}_{k'_n\in K'}$ are linearly independent.

Case (ii). $\dim H_1$ is finite. Then $\dim H_0$ is infinite. Let $J = \{j_1, j_2, j_3, \dots, j_n, \dots\}$ be an infinite subset of I such that $\{f_{j_n}\}_{j_n \in J} \subset H_0$. Let $J_0 = J \cup \{k_1\}_{k_1 \in K}$. Then by hypotheses, $\{f_i\}_{i \in J_0}$ is a frame sequence with lower bound $A_{J_0} \geq A$. So

$$A\|f\|^2 \leq \sum_{i \in J_0} |\langle f, f_i \rangle|^2, \quad \text{for all } f \in [f_i]_{i \in J_0}.$$

Again, note that $\|f_{k_1}\| < A$. Therefore, by Proposition 4.6 in [3] again, $f_{k_1} \in [f_i]_{\substack{i \in J_0 \\ i \neq k_1}}$. This is a contradiction. Hence $\{f_i\}_{i \in I}$ must be a bounded frame for H.

Remark 3.8. (i) The condition that $f \neq 0$, for all $i \in I$ in Theorem 3.7 is necessary. Indeed, if $\{f_n\} \subset H$ be given by

$$f_1 = 0, \ f_n = e_{n-1}, \quad \text{for all } n \geq 2,$$

where $\{e_n\}$ is a sequence of unit orthonormal vectors in H, then for any infinite set $K = \{k_1, k_2, \dots, k_n, \dots\}$, $\{f_{k_n}\}_{k_n \in K}$ is a frame sequence with bounds equal to 1, but $\{f_n\}$ is not a bounded frame for H.

(ii) The condition $\inf A_J > 0$ in Theorem 3.7 is also necessary. Indeed, if $\{g_n\} \subset H$ and $\{h_n\} \subset H$ be given by

$$g_n = e_n, \ n \in \mathbb{N}, \text{ and } \ h_n = \frac{e_1}{2^n}, \ n \in \mathbb{N},$$

where $\{e_n\}$ is a sequence of unit orthonormal vectors in H, then $\{f_n\} = \{g_n\} \cup \{h_n\}$ is a frame for H. Note that $\inf A_J = 0$ and for any infinite set $K = \{k_1, k_2, \dots, k_n, \dots\}$, $\{f_{k_n}\}_{k_n \in K}$ is a frame sequence but $\{f_n\}$ is not a bounded frame for H.

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References

[1] R. Balan, P. G. Casazza and D. Edidin, *On signal reconstrction without phase*. Appl. Comput. Harmon. Anal. 20 (2006), 345–356.

[2] J. J. Benedetto and M. Fickus, *Finite normalized tight frames*. Frames, Adv. Comput. Math. 18 (2–4) (2003), 357 – 385.

[3] P. G. Casazza, *The art of frame theory*. Taiwanese J. Math. 4 (2) (2000), 129 – 201.

[4] P. G. Casazza, *Custom building finite frames, Wavelets, frames and operator theory*. Contemp. Math. 345, Amer. Math. Soc., Providence, RI, 2004, 61–86.

[5] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2002.

[6] O. Christensen, *On frame perturbations*. Proc. Amer. Math. Soc. 123 (1995), 1217 – 1220.

[7] I. Daubechies, A. Grossmann and Y. Meyer, *Painless non-orthogonal expansions*. J. Math. Physics 27 (1986), 1271 – 1283.

[8] R. J. Duffin and A. C. Schaeffer, *A class of non-harmonic Fourier series*. Trans. Amer. Math. Soc. 72 (1952), 341 – 366.

[9] S.J. Favier and R.A. Zalik, *On stability of frames and Riesz bases*. Appl. Comp. Harm. Anal. 2 (1995), 160 – 173.

[10] D. Gabor, *Theory of communications*. J. IEE (London) 93 (3) (1946), 429 – 457.

[11] C. Heil and D. Walnut, *Continuous and discrete wavelet transform*. SIAM Review 31 (1989), 628 – 666.

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