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The Iteration Technique in Elementary Functional Analysis

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Abstract

In the present article we single out a pattern of arguments in elementary functional analysis, called the iteration technique here, which can enhance the teaching of some topics in this subject, especially the open mapping theorem.

1 Introduction

Slick arguments in mathematics are delightful, but also appear to be unmotivated. Students may find them too challenging instead of enjoyable, worrying about being asked in examinations. A good teacher, or a book for beginners, should try to implement tricky arguments with helpful remarks, making them psychologically more acceptable, or better, try to clarify the pattern of such an argument, in order to illuminate the idea and facilitate a learner. In the present article we single out a pattern of arguments in elementary functional analysis, called the iteration technique here, which can enhance the teaching of some topics in this subject, especially the open mapping theorem.

2 Main Results

We start with the simplest set-up for the iteration technique:

$$B \subseteq S + tB. \tag{1}$$

Here B is a bounded set in a normed linear space (or, more generally, a topological vector space) X, t is a scalar with $|t| < 1$, and S is any set in X. We assert that every element in B can be expressed as the sum of a convergent series $\sum_{k=0}^{\infty} t^n x_n$ with $x_n \in S$ for all n. Indeed, given $b \in B$, we define a sequence of points b_n $(n \geq 0)$ in B iteratively by $b_0 = b$ and $b_n = x_n + tb_{n+1}$, where x_n is some element in S. Then, for each n,

$$b = x_0 + tx_1 + t^2 x_2 + \cdots + t^{n-1} x_{n-1} + t^n b_n.$$

Since b_n is in the bounded set B and $|t| < 1$ as $n \to \infty$, the sequence $\lim t^n b_n = 0$. Done.

As an application, we show that if a normed linear space (or a topological vector space) X has an open neighbourhood U of 0 with a compact closure $\overline{U}$, then X is finite dimensional. Indeed, by reducing the open cover $\{x + \frac{1}{2}U\}_{x\in\overline{U}}$ of $\overline{U}$ to a finite subcover, we find a finite set F in X such that $\overline{U} \subseteq F + \frac{1}{2}U$ and hence $\overline{U} \subseteq F + \frac{1}{2}\overline{U}$. Here $\overline{U}$ is compact and hence is bounded. Our previous discussion shows that each point in $\overline{U}$ can be written as a convergent series $\sum_{n=1}^{\infty}(1/2^n)x_n$, where each x_n is a point in the finite set F. This shows that $\overline{U}$ is in the linear span M of F. Since U is an open neighbourhood of 0, X is contained in the finite dimensional space M.

The same working principle may apply if we modify (1) above and allow B and S to vary. We illustrate this by showing

Theorem 1 [1; P. 5]. *a set B in a Banach space (or Fréchet space) X is relative compact if and only if there is a squence $\{a_n\}$ in X converging to zero such that K is contained in its closed convex hull $\overline{\text{co}}\,\{a_n\}$.*

Proof: The "if" part is straightforward to check. Assume that K is compact. Let $\{V_n\}$ be a countable base of closed neighbourhoods of 0 with $V_n \supseteq V_{n+1}$ for all n. (In the case we are considering, i.e. X is a Banach space, we may let $V_n = \{x \in X|\ \|x\| \leq 1/n\}$.) Now we construct a sequence of compact sets K_n and a sequence of finite sets F_n with $F_n \subseteq K_n \subseteq V_n$ such that $K_0 = K$ and

$$K_n \subset F_n + \frac{1}{2}K_{n+1} \tag{2}$$

for $n \geq 0$. Suppose that K_n has been constructed. By a compactness argument, we can take a finite set F_n in K_n such that $K_n \subseteq F_n + \frac{1}{2}V_{n+1}$. It is easy to check that

$$K_n \subseteq F_n + \frac{1}{2}\,[V_n \cap 2(K_n - F_n)].$$

Let $K_{n+1} = V_{n+1} \cap 2(K_n - F_n)$, which is a compact set. Given an element x in $K \equiv K_0$, we can start the "iteration machine" to produce a sequence $\{x_n\}$ such that $x_n \in F_n$ for all n and $x = \sum_{n=0}^{\infty} (1/2)^n x_n$. Arranging all finite sets $2F_n$ $(n = 0, 1, 2, \dots)$ into a sequence $\{a_n\}$, we have $x \in \overline{\text{co}}\,\{a_n\}$. ∎

From the above argument we observe that the "only if" part of the theorem can be improved. Instead of $K \subseteq \overline{\text{co}}\,\{a_n\}$, we conclude that K is contained in the *ideal convex hull* of $\{a_n\}$. Here we explain the meaning of ideal convexity, a concept due to Lifshits (see [3], P. 138). We say that a set A in a Banach space (or a topological vector space) X is ideally convex if the sum of the series $\sum_{n=1}^{\infty} \lambda_n x_n$ is in A, whenever this series converges, for all sequences $\{x_n\}$ in A and all nonnegative real numbers λ_n with $\sum_{n=1}^{\infty} \lambda_n = 1$. If A here is bounded, the series $\sum_{n=1}^{\infty}\lambda_n x_n$ is indeed convergent. Let

$$s(\ell^1) = \{\{\lambda_n\} \in \ell^1 |\ \lambda_n \geq 0 \text{ for all } n \geq 0 \ \text{ and } \ \textstyle\sum_{n=1}^{\infty}\lambda_n = 1\},$$

the standard simplex of ℓ^1. The ideal convex hull of a bounded sequence $\{a_n\}$, denote by $\text{ico}\,\{a_n\}$, is the set consisting of all sums of the form $\sum_{n=1}^{\infty} \lambda_n a_n$. We may write $\text{ico}\,\{a_n\} = T(s(\ell^1))$, where $T\colon \ell^1 \to X$ is the bounded linear map defined by $T(\{\lambda_n\}) = \sum_{n=1}^{\infty} \lambda_n a_n$. Since $s(\ell^1)$ is ideally convex, it is not hard to see that $\text{ico}\,\{a_n\}$ is ideally convex as well. This observation does not contribute very much to Theorem 1, but the idea will become crucial in the rest of the present article.

Theorem 2. *If $\{a_n\}$ is a dense sequence in a bounded set B of a Banach space X, then the interior of B is contained in the ideal convex hull of $\{a_n\}$: $B^o \subseteq \text{ico}\,\{a_n\}$.*

Proof: (Let us recall that the interior S^0 of a set S in a topological space is the largest open set contained in S.) When B^o is empty, there is nothing to prove. Let $x \in B^o$ and let V be a bounded open neighbourhood of x such that $x + V \subseteq B$. Notice that, for each positive integer n, the sequence $\{a_k\}_{k\geq n} \equiv \{a_n,\, a_{n+1},\, \dots\}$ is still dense in B. Hence, for all n and $\varepsilon > 0$,

$$V \subseteq \{a_k\}_{k\geq n} + \varepsilon V,$$

which is our mechanism for iteration. Inductively, we can get a sequence $\{x_k\}$ in V and a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $x_0 = 0$ and $x + x_k = a_{n_k} + \varepsilon x_{k+1}$

for all $k \geq 0$. An easy computation by cancelling x_k shows $\left(\sum_{k=0}^{\infty} \varepsilon^k\right) x = \sum_{k=0}^{\infty} \varepsilon^k a_{n_k}$. Now the theorem is clear. ∎

Suppose that X is a separable Banach space. Then we may take a dense seqeunce $\{a_n\}$ in the unit ball of X. Define the bounded linear map $T\colon \ell^1 \to X$ by putting $T\{\lambda_n\} = \sum \lambda_n a_n$. By the above theorem, we see that the origin of X is in the interior of $\mathrm{ico}\{a_n\} = T(s(\ell^1))$ and hence T is surjective. We have shown the following theorem of Banach and Mazur (see [4], P. 113): *Every separable Banach space is the range of a bounded linear surjection from ℓ^1.*

The following basic property of ideally convex sets due to Lifshits, in the same vein as the last theorem, deserves to be more widely known:

Theorem 3 (see [3], P.139). *If A is an ideally convex set in a Banach space (or Fréchet space) X, then A and its closure $\overline{A}$ have the same interior: $A^o = \overline{A}^o$.*

(Recall that the interior S^o of a set S in a topological space X is the largest open set contained in S.)

Proof: Clearly $A^o \subseteq \overline{A}^o$. We assume that $U \equiv \overline{A}^o$ is nonempty; otherwise there is nothing to prove. Also, we assume $0 \in U \equiv \overline{A}^o$, because we may adjust A by a suitable translation. Let $\{V_n\}$ be a countable base of open neighbourhoods of 0 with $V_n \supseteq V_{n+1}$ for all n. Let x be a point in $\overline{A}^o$. We may take a small enough $\eta > 0$ such that $(1+\eta)x$ is still in $\overline{A}^o$. Let $\varepsilon > 0$ be such that $\eta = \varepsilon/(1-\varepsilon)$. An elementary fact in functional analysis (which is easy to check) says that if V is a neighbourhood of 0 in X, then $\overline{S} \subseteq S + V$. Then we have $\overline{A} = A + \varepsilon(U \cap V_n)$, which gives

$$U \subseteq A + \varepsilon(U \cap V_n)$$

for all n, which is the inclusion relation giving us an iteration mechansim. By induction, we can find a sequence $\{a_n\}$ in A and $\{x_n\}$ in U such that $x_0 = (1+\eta)x$, $x_n \in V_n$ for all n and $x_n = a_n + \varepsilon x_{n+1}$. It is easy to check that the sum $\sum_{n=0}^{\infty} \varepsilon^n a_n$ converges to $x_0 \equiv (1+\eta)x$. The rest is clear. ∎

Lifshit's proof of the open mapping theorem, according to Holmes [3], takes the following three steps. First step: given a bounded linear map T from a Banach space onto another, show that $\overline{T(B_X)}$ has an interior point via a categorical argument in the usual way; here B_X is the unit ball in X. Second step: check that $T(B_X)$ is ideally convex. Last step : apply the above theorem to conclude that $T(B_X)$ has an interior point. Here we make an improvement by observing that the second step also works for (closed) unbounded maps. Let T be a linear map with its domain $\mathcal{D}$, which is a dense linear subspace of a Banach space X, into another Banach space Y. Recall that the graph $\mathrm{Gr}(T)$ of T consists of all points of the form (x, Tx) in the product $X \times Y$, with $x \in \mathcal{D}$. Assume that T is closed in the sense that its graph $\mathrm{Gr}(T)$ is a closed subspace of $X \times Y$, in other words, whenever $\{x_n\}$ is a sequence in $\mathcal{D}$ with $\lim x_n = x$ and $\lim Tx_n = y$, we have $x \in D$ and $Tx = y$. With the notation given here, we make the following observation:

Lemma 4. *The set $T(\mathcal{D} \cap B_X)$ in Y is ideally convex.*

Proof: Actually, we are going to prove a stronger result: if B is a bounded ideally convex set in X, then $T(\mathcal{D} \cap B)$ is an ideally convex set in Y; (here both X and Y are allowed to be arbitrary topological vector spaces). Take a sequence $\{y_n\}$ in $T(\mathcal{D} \cap B)$ and a sequence $\{\lambda_n\}$ in $s(\ell^1)$ and assume the convergence of the series $\sum_{n=0}^{\infty} \lambda_n y_n$. We have to show that the sum of this series, say y, is also in $T(\mathcal{D} \cap B)$. For each n, there is an element x_n in $\mathcal{D} \cap B$ such that $Tx_n = y_n$. Since the sequence $\{x_n\}$ is in B, the series $\sum_n \lambda_n x_n$ is convergent, say, to x, which is in B in view of ideal convexity of B. By the closedness of T, we see that $x \in \mathcal{D}$ and $y = Tx$. Therefore $y \in T(\mathcal{D} \cap B)$. ∎

Using the above lemma and Theorem 3, together with a categirical argument, it is not hard to show that, *if T is a closed linear map from a dense domain $\mathcal{D}$ in X to Y such that $T(\mathcal{D}) = Y$, then T is open in the sense that $T(\mathcal{D} \cap U)$ is an open set in Y for all open set U in X.*

We should briefly mention that there is a general open mapping theorem valid for continuous linear maps on so-called *Ptak spaces*; see [5], Ch. IV, §8. Also, there is a well-known technique called moving hump argument which was discovered at the early development of Banach space theory. We have no intention to give some details of these topics here because they are usually not included in a course of elementary functional analysis.

References

[1] J. Diestel, Sequence Spaces and Series in Banach Spaces, Graduate texts in Math., **92**, Springer Verlag, New York, 1984.

[2] S. Goldberg, Unbounded Linear Operators, Theory and Applications, McGraw-Hill, New York, 1966; Dover Edition, New York, 1985.

[3] R.B. Holmes, Geometric Functional Analysis and its Applications, Graduate Texts in Math., **24**, Springer Verlag, New York, 1975.

[4] R. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Math., **183**, Springer Verlag, New York, 1998.

[5] Schaefer (with M.P. Wolff), Topological Vector Spaces, Graduate Texts in Math., **3**, 2nd Edition, Springer Verlag, 1999.

[6] 6 C. Swartz, Infinite Matrices and the Gliding Hump, World Science, Singapore, 1996.

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