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Classification of Integral Curves in $(2n + 1)$ Dimensional Vector Field

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Abstract

Recently Taleshian [1] derived integral curve of vector field X in E^{2n+1}. Earlier the Realisation $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ proposed by Karger and Novak [2] for vector field X in E^3. The skew-symmetric matrix and A considered in [1] has different structure from the one used in [2]. In this paper the original form of matrix A, by Karger and Novak [2] applicable in E^3 is considered and extended to be used in E^{2n+1}. It is proved that for rank$[AC] = 2n + 1$, we have Hyperhelices. and for rank$[AC] = 2n$, we have circles in Hyperplanes. However for series of parallel Hyperlines only exist when we have rank$[AC] = 1$.

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1 Introduction

The integral curves of a linear vector field on E^{2n+1} are dependent on the rank of the matrix which defines the linear vector fields. They are circles or helices in the cases when the matrix of the linear vector field has even or respectively odd rank. In recent years the theory of helices in higher dimensions has been extensively studied. In the present paper, we investigate the theory of integral curves of a linear vector field and show that the theory of integral curves of a linear vector field in the (2n+1)-dimensional Euclidean space($n \geq 1$) is the same as in the case $n = 1$ [1].

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2 Preliminaries

Let $\alpha : I \to E^n$, $t \to \alpha(t)$ $(t \in I)$ be a parametrized curve and let X be a vector field in E^n ([1],[2]). If $\frac{d\alpha}{dt} = X(\alpha(t))$, holds true, then the curve α is called an integral curve of the vector field X. Let V be a vector space over R of dimension $2n+1$. A vector field X on V is called linear if $X(v) = A(v)$, each $v \in V$, where A is a linear mapping from V into V. Let$A \in R^{2n+1}_{2n+1}$ be a skew-symmetric matrix. Then we can choose an orthonormal basis $\varphi \in R^{2n+1}$, such that the matrix A reduces to the form

$$\begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & 0\dots & 0 & 0 & 0 \\ -k_1 & 0 & 0 & 0 & k_2 & 0\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 0\dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0\dots & 0 & k_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\dots & -k_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\dots & 0 & 0 & 0 \end{bmatrix} \in R^{2n+1}_{2n+1} \tag{1}$$

$k \in R - \{0\}$. If $C \in R^{2n+1}_1$,

$$C =^t (a_1, a_2, a_3, ..., a_{2n}, a_{2n+1}) \tag{2}$$

is a column matrix, then we showed that the value of X at any point$P$ of E^{2n+1} can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P \\ 1 \end{bmatrix}, \tag{3}$$

where $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ is called the matrix of the linear vector field$X([1],[2])$.

3 Linear vector fields and Integral curves in E^3

Let X be a linear vector field in E^3 and let $\{o; u_1, u_2, u_3\}$ be an orthonormal frame of E^3; then the matrix in this frame can be written as

$$\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & a \\ -k & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, rank[AC] = 3, [2]. \tag{4}$$

Then the value of X at a point $P = (x, y, z)$ of E^3 is

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & 0 & a \\ -k & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{5}$$

or

$$X(P) = (ky + a, -kx + b, c). \tag{6}$$

On the other hand if the curve $\alpha : I \to E^3, t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ is an integral curve of X, then denoting by dot the derivative w.r.t. the variable t, we can write the differential equation

$$\alpha^{\cdot}(t) = X(\alpha(t), \forall t \in I \tag{7}$$

as the system of differential equations

$$x^{\cdot} = ky + a, y^{\cdot} = -kx + b, z^{\cdot} = c \tag{8}$$

For the shake of shortness,we set $k = 1$ and then (8) reduces to

$$x^{\cdot} = y + a, y^{\cdot} = -x + b, z^{\cdot} = c \tag{9}$$

Then the solution of the last equation of this system is

$$z = ct + d \tag{10}$$

For the solutions of first two equations we derivate the second equation and obtain that

$$y^{\cdot\cdot} + y = -a, \tag{11}$$

which is the first order linear differential equation with constant coefficent. We know the solution of this equation is

$$y = A\cos t + B\sin t - a \tag{12}$$

On the other hand the derivation of (11) and the second of (12) give us that

$$x = A\sin t - B\cos t + b. \tag{13}$$

Thus the integral curves of X can be written as

$$\alpha(t) = (A\sin t - B\cos t + b, A\cos t + B\sin t - a, ct + d). \tag{14}$$

This is a family of inclined curves with common axes and the same parameter, since we have

$$H = \frac{K_1}{K_2} = \frac{1}{c}\sqrt{A^2 + B^2}, \tag{15}$$

where K_1 and K_2 are the curvatures of the curve and H is constant for each one of the cures. Now we assume the case that rank [AC]=2. In this case $c = 0$ since we know that $k \neq 0$. Hence (13) gives us the equation of integral curves as

$$\alpha(t) = (A\sin t - B\cos t + b, A\cos t + b\sin t - a, d). \tag{16}$$

The curves are the circle each one of which lies on the parallel planes and the centers of these circle are located on an axis perpendicular to those parallel planes. Finally, assume that rank$[AC] = 1$. In this case we have that$\lambda = 0$ and the system (16) reduces to the system

$$x^{\cdot} = a, y^{\cdot} = b, z^{\cdot} = c. \tag{17}$$

Then the solution of this system is

$$\alpha(t) = (at + d_1, bt + d_2, ct + d_3). \tag{18}$$

These integral curves are parallel straight lines ([1],[2]).

4 The Normal Form Case

The normal form of the skew-symmetric matrix A is

$$A = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & 0\dots & 0 & 0 & 0 \\ -k_1 & 0 & 0 & 0 & k_2 & 0\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 0\dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0\dots & 0 & k_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\dots & -k_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\dots & 0 & 0 & 0 \end{bmatrix} \in R^{2n+1}_{2n+1} \tag{19}$$

$k \in R - \{0\}$. In this case we can prove the following theorem:

Theorem 4.1. Let$X$ be a linear vector field in the (2n+1)-dimensional Euclidean space determined by the matrix $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ with respect to an orthonormal frame $\{O; u_1, u_2, ..., u_{2n+1}\}$, whose A is normal formed skew-symmetric matrix, C is a column matrix. Then the integral curves of X have the following properties:
(i) If rank$[AC] = 2\lambda + 1, 1 \leq \lambda \leq n$, then these curves are same parametrized circular helices which have a common axis.
(ii) If rank$[AC] = 2\lambda$, $1 \leq \lambda \leq n$, then those curves are circles in parallel planes whose centeres lie on a same straight line perpendicular to those planes.

(iii) If $\text{rank}[AC] = 1$, then these curves are the parallel straight lines.

Proof : Let X a linear vector field for all points $P = (x_1, ..., x_{2n+1}) \in E^{2n+1}$. Then we have

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P \\ 1 \end{bmatrix}, \tag{20}$$

or
$X(P) = (k_1x_2 + a_1, -k_1x_1 + a_2, ..., k_nx_{2n} + a_{2n-1}, -k_nx_{2n-1} + a_{2n}, a_{2n+1})$.

Moreover, if a curve $\alpha : I \subset R \to E^{2n+1}$ is an integral curve of the vector field X , then we can write that

$$\frac{d\alpha}{dt} = X(\alpha(t)). \tag{21}$$

The integral curve, with the initial condition $\alpha(t) = P$ and $P = (x_1, ..., x_{2n+1})$ is a solution curve of the differential equation

$$\frac{d\alpha}{dt} = X(P) \tag{22}$$

which means that

$$\begin{aligned} &\frac{d\alpha_1}{dt} = x_2 + a_1, \frac{d\alpha_2}{dt} = -x_1 + a_2, ..., , \frac{d\alpha_{2n-1}}{dt} = x_{2n} + a_{2n-1}, \\ &\frac{d\alpha_{2n}}{dt} = -x_{2n-1} + a_{2n}, \frac{d\alpha_{2n+1}}{dt} = a_{2n+1} = constant. \end{aligned} \tag{23}$$

This means that $k_i = 1$, and $1 \leq i \leq n$. If we solve the differential equation

$$\frac{d\alpha_{2n+1}}{dt} = c \tag{24}$$

we get

$$\alpha_{2n+1} = ct + d. \tag{25}$$

The other $2n$ equations can be solved in pairs. For example let us solve the first two equations are

$$x_2 = A_1 \cos t + B_1 \sin t - a_1, x_1 = A_1 \sin t + B_1 \cos t + a_2 \tag{26}$$

Continuing in this way, we get

$$x_{2n-1} = A_n \sin t - B_n \cos t + a_{2n}, x_{2n} = A_n \cos t + B_n \sin t + a_{2n-1}. \tag{27}$$

Using these solutions the expreession of $\alpha(t)$ can be written as

$$\begin{aligned}\alpha(t) &= (A_1 \sin t - B_1 \cos t + a_2, A_1 \cos t + B_1 \sin t - a_1, ..., \\ & A_n \sin t - B_n \cos t + a_{2n}, A_n \cos t + B_n \sin t - a_{2n-1}, ct + d). \quad (28)\end{aligned}$$

Now, we that $\alpha(t)$ is an helx in E^{2n} and we find the axis of it. In order to show this, we must show that [4]

$$H_1 = \frac{K_1}{K_2} = constant. \quad (29)$$

The vectors $\alpha^{\cdot}$, $\alpha^{\cdot\cdot}$, $\alpha^{\cdot\cdot\cdot}$ and α^4 ,..., α^{2n+1} are liearly dependent. Hence there exists two curvatures K_1 and K_2. Since the curve $\alpha(t)$ does not have the unit velocity, we must apply the change of parameter to normalize the velocity.

$$\alpha^{\cdot} = (A_{1\cos t} + B_1 \sin t, -A_1 \sin t + B_1 \cos t, ..., A_n \cos t + B_n \sin t, -A_n \sin t + B_n \cos t, c) \quad (30)$$

and if we denote

$$\sum_{i=1}^{n} (A_1^2 + B_1^2) \quad (31)$$

by Y we have

$$\|\alpha^{\cdot}\| = \sqrt{Y + c^2}, s = \int_0^t \|\alpha^{\cdot}(t)\| dt \quad (32)$$

and if we denote $\sqrt{Y + c^2}$ by γ we get $t = \frac{s}{\gamma}$. Since $E_1(s) = \beta^{\cdot}(s)$ and let be $<, >$, inner product operator, then

$$< \beta^{\cdot}(s), \beta^{\cdot\cdot}(s) >= 0, < \beta^{\cdot\cdot\cdot}(s), \beta^{\cdot\cdot}(s) >= 0. \quad (33)$$

$$E_2(s) = \beta^{\cdot\cdot}(s), E_3(s) = \frac{1}{\gamma^4}(\beta^{\cdot\cdot\cdot}(s) + Y, E_1(s)) \quad (34)$$

and from [4] we have

$$K_i(s) = \frac{\|E_{i+1}(s)\|}{\|E_i(s)\|}. \quad (35)$$

we have

$$H_1(s) = \sqrt{\frac{Y}{C}} \quad (36)$$

This means that curve $\beta(s)$ and so the curve $\alpha(t)$ are helices. Let U be the axis of $\alpha(t)$. Since $U \in S_p\{V_1, V_3\}$, we have

$$U = V_1 \cos\varphi + V_3 \sin\varphi. \quad (37)$$

where V_1 and V_3 are the Frenet Vector fields of the curve. Since

$$V_1 = \frac{E_1(s)}{\|E_1(s)\|}, V_3 = \frac{E_3(s)}{\|E_3(s)\|} \quad (38)$$

$$H_1(s) = \sqrt{\frac{Y}{C}} \tan \varphi. \tag{39}$$

and

$$\varphi = \arctan \sqrt{\frac{Y}{C}} \tag{40}$$

$$U = \frac{C}{\alpha} V_1 + \frac{\sqrt{Y}}{\alpha} V_3. \tag{41}$$

By combining(38)and (41) we get

$$U = (0, 0, ..., 0, 1). \tag{42}$$

The curve $\alpha(t)$ is a circular helix because if the curve $\alpha(t)$ is translated by T , where

$$T = (-a_1, a_2, -a_3, a_4 - +, ..., a_{2n}, 0). \tag{43}$$

We obtain

$$a_1^2 + a_2^2 + ... + a_{2n}^2 = \sum_{i=1}^{n} (A_i^2 + B_i^2) = r = constant. \tag{44}$$

This completes the proof of the first part of the theorem.

(ii) Let rank$[AC] = 2\lambda$, $1 \leq \lambda \leq n$ then:

a) if rank$[AC] = 2n$, $\lambda = n$ then dfferential equation system becames,

$$\frac{dx_1}{dt} = x_2 + a_1, \frac{dx_2}{dt} = -x_1 + a_2, ...,$$

$$\frac{dx_{2n-1}}{dt} = x_{2n} + a_{2n-1}, \frac{dx_{2n}}{dt} = -x_{2n-1} + a_{2n}, \frac{dx_{2n+1}}{dt} = 0. \tag{45}$$

This system of differential equations has the solution

$$\begin{aligned} \alpha(t) \;\; = \;\; & (A_1 \sin t - B_1 \cos t + a_2, A_1 \cos t + B_1 \sin t - a_1, ..., \\ & A_n \sin t - B_n \cos t + a_{2n}, A_n \cos t + B_n \sin t - a_{2n-1}, d) \end{aligned} \tag{46}$$

Itis trivial to show that the curve $\alpha(t)$ are circles.

b) Let rank$[AC] = r$, $r = 2, ..., 2n - 2$, in this case

$$Rank[AC] = r \Leftrightarrow k_i = 0, \frac{r}{2} + 1 \leq i \leq n. \tag{47}$$

So that

$$\frac{dx_1}{dt} = x_2 + a_1, \frac{dx_2}{dt} = -x_1 + a_2, ...,$$

$$\frac{dx_{r-1}}{dt} = x_r + a_{r-1}, \frac{dx_r}{dt} = -x_{r+1} + a_r, \frac{dx_j}{dt} = 0, r+1 \leq j \leq 2n+1. \quad (48)$$

Therefore in this case $\alpha(t)$ is

$\alpha(t) = (A_1 \sin t - B_1 \cos t + a_2, A_1 \cos t + B_1 \sin t - a_1, ..., A_{\frac{r}{2}} \sin t - B_{\frac{r}{2}} \cos t + a_r, A_{\frac{r}{2}} \cos t + B_{\frac{r}{2}} \sin t - a_{r-1}, d_{r+2}, d_{r+3}, ..., d_{2n+1})$.

Again the curves $\alpha(t)$ are circles.

Rank$[AC] = 2\lambda + 1, \leq \lambda \leq n$.
a) If rank$[AC] = 2\lambda + 1$, $\lambda = n$
$\alpha(t)$ is same as the first part of the theorem.
b) Let rank$[AC] = 2\lambda + 1 = r + 1$, $r = 2, 4, ..., 2n - n$,

then rank$[AC] = r + 1 \Leftrightarrow \lambda_i = 0, a_{r+1} \neq 0, r + 1 \leq i \leq n$.
Hence

$$\dot{x_1} = x_r + a_1, \dot{x_2} = -x_1 + a_2, ...,$$

$$\dot{x_{r-1}} + a_{r-1}, \dot{x_r} = -x_{r-1} + a_r, \dot{x_{r+1}} = a_{r+1}, \dot{x_j} = 0, r+2 \leq j \leq 2n+1. \quad (49)$$

The solution of this system is

$\alpha(t) = (A_1 \sin t - B_{1 \cos t} + a_2, A_1 \cos t + B_1 \sin t - a_1, ..., A_{\frac{r}{2}} \sin t - B_{\frac{r}{2}} \cos t + a_r, A_{\frac{r}{2}} \cos t + B_{\frac{r}{2}} \sin t - a_{r-1}, a_{r+1}t + d + d_{r+2}, ..., d_{2n+1}$.

Obviously $\alpha(t)$ are again circular helices.

4. If rank$[AC] = 1$. Then $\lambda_i = 0$ which gives us a system of the differential equations. This system of differential equations has the solution $\alpha(t)$ which are are parallel straight lines,in E^{2n+1}.

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