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# Fuzzy Programming Based on Interval-Valued Fuzzy Numbers and Ranking

**Jin-Shieh Su**

Department of Applied Mathematics
Chinese Culture University
Hwakang, Yangminshan, Taipei, Taiwan, R.O.C.

**Abstract**

In this paper, we use interval-valued fuzzy numbers to fuzzify the crisp linear programming to three cases. The first case, we use interval-valued fuzzy numbers to fuzzify the coefficients in the objective function. We get a linear programming in the fuzzy sense. The second case, we use interval-valued fuzzy numbers to fuzzify the coefficients $a_{kj}$ in the constraints about $x_j, j = 1, 2, \cdots, n$ and the constants $b_k, k = 1, 2, \cdots, m$. We also get a linear programming in the fuzzy sense. The third case, we combine the first and the second cases.

**Keywords:** Fuzzy programming, fuzzy objective function, interval-valued fuzzy numbers, fuzzy sense

# 1 Introduction

In paper [1,6,8], they use fuzzy number to fuzzify the crisp linear programming. They do not use interval-valued fuzzy numbers to fuzzify. In [1], for crisp linear programming, the constraints equations are $\sum_{j=1}^{n} a_{kj}x_j \leqq b_k, k = 1, 2, \cdots, m$. They use fuzzy number $\widetilde{a}_{kj}, \widetilde{b}_k$ to fuzzify and get the fuzzy numbers inequality $\sum_{j=1}^{n} \widetilde{a}_{kj}x_j \precsim \widetilde{b}_k,\ k = 1, 2, \cdots, m$. Then they use ranking of fuzzy numbers to get linear programming in the fuzzy sense. They do not defuzzify the objective function and did not use interval-valued fuzzy numbers to defuzzify. In [6], they use trapezoidal fuzzy numbers to fuzzify $c_j, a_{kj}, b_k$ as $\widetilde{c}_j, \widetilde{a}_{kj}, \widetilde{b}_k$. Then get $\widetilde{Z} = \sum_{j=1}^{n} \widetilde{c}_j x_j$, $\sum_{j=1}^{n} \widetilde{a}_{kj}x_j \precsim \widetilde{b}_k, k = 1, 2, \cdots, m$. They reduce it to linear programming in the fuzzy sense. They do not use interval-valued fuzzy numbers to discuss. In this paper, we use interval-valued fuzzy numbers to consider this problem. In §2, we discuss interval-valued fuzzy numbers and their ranking which will be used in §3, 4. In §3, for crisp linear programming under constraints is as following: $\sum_{j=1}^{n} a_{kj}x_j \leqq b_k, k = 1, 2, \cdots, m, x_j \geqq 0, j = 1, 2, \cdots, n$ we find optimal solution of objective function $Z = \sum_{j=1}^{n} c_j x_j$. In monopoly market, the price $c_j, j = 1, 2, \cdots, n$ can be determined by the factory. If $a_{kj}, b_k, j = 1, 2, \cdots, n, k = 1, 2, \cdots, m$ do not vary in the plan period $T$, but $c_j$ in the plan period $T$ for a perfect competitive market may fluctuate a little, we need to fuzzify $c_j$ to $\widetilde{c}_j$. In this plan period $T$, the grade of membership is not necessary equal to 1. We suppose that the grade of membership lies in the interval $[\lambda, 1], 0 < \lambda < 1$. We set $\widetilde{c}_j$ to be level $(\lambda,1)$ i-v fuzzy number. Through this, we get the linear programming in the fuzzy sense. This is stated in theorem 1. In §3.3, we fuzzify $a_{kj}$ and $b_k, j = 1, 2, \cdots, n, k = 1, 2, \cdots, m$. in the constraints for the crisp linear programming to interval-valued fuzzy numbers $\widetilde{a}_{kj}, \widetilde{b}_k$ and get $\sum_{j=1}^{n} \widetilde{a}_{kj}x_j \precsim \widetilde{b}_k, k = 1, 2, \cdots, m$. Using ranking of the interval-valued fuzzy numbers in §2, we have linear programming in the fuzzy sense. This is stated in theorem 2. In §3.4, we combine theorem 1 and 2 and obtain fuzzy objective function $\widetilde{Z} = \sum_{j=1}^{n} \widetilde{c}_j x_j$ and fuzzy constraints $\sum_{j=1}^{n} \widetilde{a}_{kj}x_j \precsim \widetilde{b}_k, k = 1, 2, \cdots, m$. Then we have a linear programming in the fuzzy sense. This is stated in theorem 3. In §4, we give an example and §5 we give the discussions.

## 2 Interval-Valued Fuzzy Numbers and Ranking

For the purpose to consider fuzzy programming based on interval-valued fuzzy numbers and ranking, we first consider the following:

**Definition 1** $\widetilde{a}$ is called a fuzzy point, if its membership function on $R$=$(-\infty, +\infty)$ is

$$\mu_{\widetilde{a}}(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases} \quad (1)$$

**Definition 2** $\widetilde{C}$ is called a level $\lambda$ fuzzy number, $0 < \lambda \leqq 1$, if its membership function is

$$\mu_{\widetilde{C}}(x) = \begin{cases} \frac{\lambda(x-a)}{b-a} & a \leqq x \leqq b \\ \frac{\lambda(c-x)}{c-b} & b \leqq x \leqq c \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We denote $\widetilde{C} = (a, b, c; \lambda)$. When $a = b = c, \lambda = 1$ then $(a, a, a; 1) = \widetilde{a}$ is a fuzzy point.

**Definition 3** A fuzzy set is called the level $\alpha$ fuzzy interval, $0 \leqq \alpha \leqq 1$ and denote it by $[a, b; \alpha], a < b$, if its membership function is

$$\mu_{[a,b;\alpha]}(x) = \begin{cases} \alpha & a \leqq x \leqq b \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

When $a = b, \alpha = 1$ then $[a, a; 1] = \widetilde{a}$ is a fuzzy point.

**Definition 4** ([3]) An interval-valued fuzzy set (i-v fuzzy set for short) $\widetilde{A}$ on $R$ is given by $\widetilde{A} \triangleq \{(x, [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)])\},\ x \in R$ where $\mu_{\widetilde{A}^L}$ and $\mu_{\widetilde{A}^U}$ maps $R$ into $[0, 1]$ and $\mu_{\widetilde{A}^L} \leqq \mu_{\widetilde{A}^U}, \forall x \in R$. Denote $\bar{\mu_{\widetilde{A}}}(x) = [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)], x \in R$ or

$$\widetilde{A} = [\widetilde{A}^L, \widetilde{A}^U] \tag{4}$$

Then the grade of membership of i-v fuzzy set $\widetilde{A}$ at $x$ belongs to the interval $[\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)]$

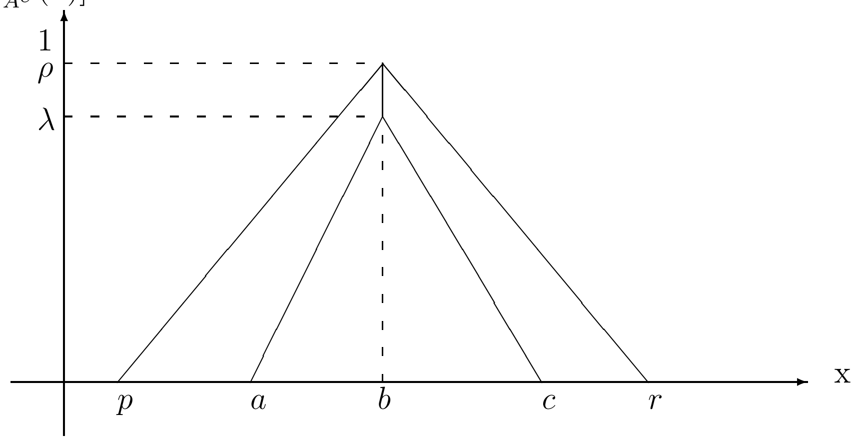


Fig.1 level $(\lambda, \rho)$ i-v fuzzy number

Let

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \frac{\lambda(x-a)}{b-a} & a \leqq x \leqq b \\ \frac{\lambda(c-x)}{c-b} & b \leqq x \leqq c \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

then $\widetilde{A}^L = (a, b, c; \lambda)$ is called level $\lambda$ fuzzy number.

Let

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} \frac{\rho(x-p)}{b-p} & p \leqq x \leqq b \\ \frac{\rho(r-x)}{r-b} & b \leqq x \leqq r \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

then $\widetilde{A}^U = (p, b, r; \rho)$, where $0 < \lambda \leqq \rho \leqq 1, p < a < b < c < r$. We get i-v fuzzy set $\widetilde{A} \triangleq \{(x, [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)])\}, x \in R$. Denote $\widetilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)] = [\widetilde{A}^L, \widetilde{A}^U]$ and call $\widetilde{A}$ level $(\lambda, \rho)$ i-v fuzzy number.

The family of all level $(\lambda, \rho)$ i-v fuzzy numbers is defined as following, where $0 < \lambda \leqq \rho \leqq 1$,

$$F_{IN}(\lambda, \rho) = \{[(a, b, c; \lambda), (p, b, r; \rho)] | p < a < b < c < r, p, a, b, c, r \in R\} \quad (7)$$

Let $\widetilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)] = [\widetilde{A}^L, \widetilde{A}^U] \in F_{IN}(\lambda, \rho)$. From eqs.(5) and (6), we get the left and right endpoint of $\alpha$-cut as following:

$$\begin{aligned} \text{if } 0 \leqq \alpha < \lambda \text{ then } & A_l^L(\alpha) = a + (b-a)\frac{\alpha}{\lambda}, \quad A_r^L(\alpha) = c - (c-b)\frac{\alpha}{\lambda} \\ & A_l^U(\alpha) = p + (b-p)\frac{\alpha}{\rho}, \quad A_r^u(\alpha) = r - (r-b)\frac{\alpha}{\rho} \quad (8) \\ \text{and if } \lambda \leqq \alpha \leqq \rho \text{ then } & A_l^U(\alpha) = p + (b-p)\frac{\alpha}{\rho}, \quad A_r^U(\alpha) = r - (r-b)\frac{\alpha}{\rho} \end{aligned}$$

Let $\widetilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] = [\widetilde{B}^L, \ \widetilde{B}^U] \in F_{IN}(\lambda, \rho)$. Through operations $\oplus$ of level $\lambda$ fuzzy numbers and level $\rho$ fuzzy numbers, we can get the following

$$\widetilde{A}^L \oplus \widetilde{B}^L = (a+d, b+e, c+g; \lambda), \qquad \widetilde{A}^U \oplus \widetilde{B}^U = (p+u, b+e, r+w; \rho) \quad (9)$$

**Definition 5** $\widetilde{A}, \widetilde{B} \in F_{IN}(\lambda, \rho), k \in R$

$$\widetilde{A} \oplus \widetilde{B} = [\widetilde{A}^L \oplus \widetilde{B}^L, \ \widetilde{A}^U \oplus \widetilde{B}^U] \quad (10)$$

$$k\widetilde{A} = [k\widetilde{A}^L, \ k\widetilde{A}^U] \quad (11)$$

From eqs.(9)$\sim$(11), we have the following

**Property 1** Let $\widetilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)], \widetilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] \in F_{IN}(\lambda, \rho)$ then

$(1^0)$ $\widetilde{A} \oplus \widetilde{B} = [(a+d, b+e, c+g; \lambda), (p+u, b+e, r+w; \rho)]$

$(2^0)$ when $k > 0$ then $k\widetilde{A} = [(ka, kb, kc; \lambda), (kp, kb, kr; \rho)]$

$(3^0)$ when $k < 0$ then $k\widetilde{A} = [(kc, kb, ka; \lambda), (kr, kb, kp; \rho)]$

$(4^0)$ when $k = 0$ then $k\widetilde{A} = [(0, 0, 0; \lambda), (0, 0, 0; \rho)]$

with the similarly arguments as [9], we use signed distance to consider ranking. In order to consider ranking of $F_{IN}(\lambda, \rho)$ in $R$, we first consider ranking on $R$.

**Definition 6** Let $a, 0 \in R$, we define the signed distance $d^*$ as $d^*(a, 0) = a$.

The meaning of $d^*$ is that when $a > 0$, $d^*(a, 0) = a > 0$, i.e. $a$ is at the right of 0 and the distance from 0 is $a$, when $a < 0$ , $d^*(a, 0) = -a < 0$, i.e. $a$ is at the left of 0 and the distance from 0 is $-a$. Therefore, $d^*(a, 0)$ is called the signed distance of $a$ from 0.

The signed distance on $F_{IN}(\lambda,\rho)$, by definition 6, can be defined by the following: if $\widetilde{A}=[(a,b,c;\lambda),(p,b,r;\rho)]=[\widetilde{A}^L,\widetilde{A}^U]\in F_{IN}(\lambda,\rho)$. The $\alpha$-level set of $\widetilde{A}=[\widetilde{A}^L,\widetilde{A}^U]\in F_{IN}(\lambda,\rho)$ is defined as $\{x\mid \mu_{\widetilde{A}^U}(x)\geqq\alpha\}-\{x\mid \mu_{\widetilde{A}^L}(x)>\alpha\}$, then by Decomposition Theorem and Fig.2 we have

$$\widetilde{A}=\bigcup_{0\leqq\alpha<\lambda}([A_l^U(\alpha),A_l^L(\alpha);\alpha]\cup[A_r^L(\alpha),A_r^U(\alpha);\alpha])\cup(\bigcup_{\lambda\leqq\alpha\leqq\rho}[A_l^U(\alpha),A_r^U(\alpha);\alpha]) \tag{12}$$

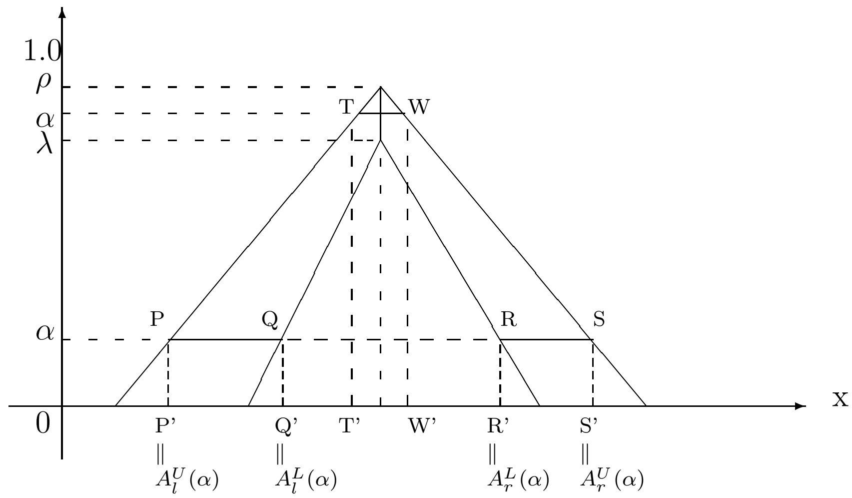


Fig.2 signed distance of i-v fuzzy numbers

We have following one-one onto mapping for each $\alpha$. When $0\leqq\alpha<\lambda$,

$$[A_l^U(\alpha),A_l^L(\alpha);\alpha](\text{corresponding PQ})\longleftrightarrow[A_l^U(\alpha),A_l^L(\alpha)]=[P',Q'],$$

$$[A_r^L(\alpha),A_r^U(\alpha);\alpha](\text{corresponding RS})\longleftrightarrow[A_r^L(\alpha),A_r^U(\alpha)]=[R',S'],$$

$$\text{and}\ \ [A_l^U(\alpha),A_l^L(\alpha)]\cap[A_r^L(\alpha),A_r^U(\alpha)]=\emptyset,$$

and when $\lambda\leqq\alpha\leqq\rho$,

$$[A_l^U(\alpha),A_r^U(\alpha);\alpha](\text{corresponding TW})\longleftrightarrow[A_l^U(\alpha),A_r^U(\alpha)]=[T',W'].$$

From definition 6, we obtain when $0\leqq\alpha<\lambda$, $d^*(A_l^U(\alpha),0)=A_l^U(\alpha)$, $d^*(A_l^L(\alpha),0)=A_l^L(\alpha)$, $d^*(A_r^L(\alpha),0)=A_r^L(\alpha)$, and $d^*(A_r^U(\alpha),0)=A_r^U(\alpha)$. That is to say, the signed distances of $P',Q',R',S'$ from 0 are $A_l^U(\alpha)$,$A_l^L(\alpha)$, $A_r^L(\alpha)$, and $A_r^U(\alpha)$. Therefore, the signed distance of interval $[A_l^U(\alpha),A_l^L(\alpha)]$ from 0 is $d^*([A_l^U(\alpha),A_l^L(\alpha)],0)$. It can be defined as

$$\begin{aligned}\frac{1}{2}[d^*(A_l^U(\alpha),0)+d^*(A_l^L(\alpha),0)] &= \frac{1}{2}[A_l^U(\alpha)+A_l^L(\alpha)]\\ &= \frac{1}{2}[a+p+(b-a)\frac{\alpha}{\lambda}+(b-p)\frac{\alpha}{\rho}]\end{aligned}$$

Similarly, $d^*([A_r^L(\alpha), A_r^U(\alpha)], 0) = \frac{1}{2}[c + r - (c-b)\frac{\alpha}{\lambda} - (r-b)\frac{\alpha}{\rho}]$
Since, [P',Q'] $\cap$ [R',S']=$\emptyset$, for the $\alpha$-cut of $\widetilde{A}$ on $0 \leqq \alpha < \lambda$, the signed distance of [P',Q'] $\cup$ [R',S'] from 0, can be defined as

$$\begin{aligned} &d^*([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)], 0) \\ &\quad = \frac{1}{2}[d^*([A_l^U(\alpha), A_l^L(\alpha)], 0) + d^*([A_r^L(\alpha), A_r^U(\alpha)], 0)] \\ &\quad = \frac{1}{4}[a + p + c + r + (2b - a - c)\frac{\alpha}{\lambda} + (2b - p - r)\frac{\alpha}{\rho}] \end{aligned}$$

This function is continuous on $0 \leqq \alpha < \lambda$ with respect to $\alpha$. It follows that, by integration, we can find the average value.

$$\begin{aligned} &\frac{1}{\lambda}\int_0^\lambda d^*([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)], 0)d\alpha \\ &\quad = \frac{1}{8}[a + c + 2b + 2p + 2r + (2b - p - r)\frac{\lambda}{\rho}] \end{aligned} \tag{13}$$

Similarly, when $\lambda \leqq \alpha \leqq \rho$,

$$\begin{aligned} d^*([A_l^U(\alpha), A_r^U(\alpha)], 0) &= \frac{1}{2}[d^*(A_l^U(\alpha), 0) + d^*(A_r^U(\alpha), 0)] \\ &= \frac{1}{2}[A_l^U(\alpha) + A_r^U(\alpha)] \\ &= \frac{1}{2}[p + r + (2b - p - r)\frac{\alpha}{\rho}] \end{aligned}$$

This function is also continuous on $\lambda \leqq \alpha \leqq \rho$ with respect to $\alpha$. By the same reason, through integration, find the average value, $\lambda < \rho$.

$$\begin{aligned} &\frac{1}{\rho - \lambda}\int_\lambda^\rho d^*([A_l^U(\alpha), A_r^U(\alpha)], 0)d\alpha \\ &\quad = \frac{1}{4}[2b + p + r + (2b - p - r)\frac{\lambda}{\rho}] \end{aligned} \tag{14}$$

From eqs.(12)$\sim$(14) we define the signed distance of $\widetilde{A}$ from $\widetilde{0}$.
**Definition 7** Let $\widetilde{A} = [a, b, c; \lambda), (p, b, r; \rho)] \in F_{IN}(\lambda, \rho)$. The signed distance of $\widetilde{A}$ from $\widetilde{0}$ is defined as
($1^0$) when $0 < \lambda < \rho \leqq 1$,

$$\begin{aligned} d(\widetilde{A}, \widetilde{0}) &= \frac{1}{\lambda}\int_0^\lambda d^*([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)], 0)d\alpha \\ &\quad + \frac{1}{\rho - \lambda}\int_\lambda^\rho d^*([A_l^U(\alpha), A_r^U(\alpha)], 0)d\alpha \\ &= \frac{1}{8}[6b + a + c + 4p + 4r + 3(2b - p - r)\frac{\lambda}{\rho}] \end{aligned}$$

($2^0$) when $0 < \lambda = \rho \leqq 1$,

$$d(\widetilde{A}, \widetilde{0}) = \frac{1}{8}[4b + a + c + p + r]$$

By definition 7, we can define the ranking of $F_{IN}(\lambda, \rho)$ as following:
**Definition 8** Let $\widetilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)]$, $\widetilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] \in F_{IN}(\lambda, \rho)$,

$$\widetilde{B} \prec \widetilde{A} \text{ iff } d(\widetilde{B}, \widetilde{0}) < d(\widetilde{A}, \widetilde{0})$$
$$\widetilde{B} \approx \widetilde{A} \text{ iff } d(\widetilde{B}, \widetilde{0}) = d(\widetilde{A}, \widetilde{0})$$

From linear order property of $(R, <, =)$ and definition 8, we get the following property.
**Property 2** Let $\widetilde{A}, \widetilde{B}, \widetilde{C} \in F_{IN}(\lambda, \rho)$.

(a) $(F_{IN}(\lambda, \rho), \approx, \prec)$ satisfies the law of trichotomy, i.e., only one of $\widetilde{A} \prec \widetilde{B}$, $\widetilde{A} \approx \widetilde{B}$, $\widetilde{B} \prec \widetilde{A}$ will occur.

(b) $(F_{IN}(\lambda, \rho), \approx, \prec)$ satisfies the following ordering relation

($1^0$) $\widetilde{A} \precsim \widetilde{A}$

($2^0$) $\widetilde{A} \precsim \widetilde{B}$ and $\widetilde{B} \precsim \widetilde{A} \Longrightarrow \widetilde{A} \approx \widetilde{B}$

($3^0$) $\widetilde{A} \precsim \widetilde{B}$ and $\widetilde{B} \precsim \widetilde{C} \Longrightarrow \widetilde{A} \precsim \widetilde{C}$

From property 2, we known that "$\approx, \prec$," is the linear order on $F_{IN}(\lambda, \rho)$.
**Definition 9** Let $\widetilde{A}_n$, $n = 1, 2, 3, \cdots$, $\widetilde{B} \in F_{IN}(\lambda, \rho)$. If $\widetilde{A}_n \precsim \widetilde{B}$ $\forall n = 1, 2, \cdots$, then we write $\widetilde{B} = \max_{n \in \{1,2,3,\cdots\}} \widetilde{A}_n$

# 3 Fuzzy objective function in linear programming based on interval-valued fuzzy numbers

### 3.1 Crisp linear programming

Consider the following crisp linear programming problem.
A factory produces $n$ productors $X_j$, $j = 1, 2, \cdots, n$. Each product requires $m$ processes $A_k$, $k = 1, 2, \cdots, m$. Product $X_j$, through process $A_k$ requires $a_{kj}$ hours, $k = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Each process $A_k$ provides $b_k$ hours, $k = 1, 2, \cdots, m$. Let the quantity produced for $X_j$ be $x_j$, $j = 1, 2, \cdots, n$. Then we get the following constraint functions

$$\sum_{j=1}^{n} a_{kj} x_j \leqq b_k, \ k = 1, 2, \cdots, m.$$

In monopoly market, the monopolist can determine the sale price $c_j(>0)$, $j=1,2,\cdots,n$ and can get total income $Z=\sum_{j=1}^{n} c_j x_j$ Therefore, we have the following crisp linear programming objective function

$$\text{Maximize} \qquad \text{Z} = \sum_{j=1}^{n} c_j x_j \tag{15}$$

subject to:

$$\sum_{j=1}^{n} a_{kj} x_j \leqq b_k, \quad k=1,2,\cdots,m \tag{16}$$

$$x_j \geqq 0, \quad j=1,2,\cdots,n \tag{17}$$

Let
$L=\{(x_1,x_2,\cdots,x_n)|\sum_{j=1}^{n} a_{kj}x_j \leqq b_k, k=1,2,\cdots,m, x_j \geqq 0, j=1,2,\cdots,n\}$.
Obvious, $L$ is a closed bounded convex set. Under condition eqs.(16) and (17), monopolist can find out $x_j$, $j=1,2,\cdots,n$ which maximize the total income Z. This is a crisp linear programming problem. We can use simplex method to find the optimal solution. Suppose that this optimal solution is the production quantity $x_j^{(0)}, j=1,2,\cdots,n$. The total income $Z_0=\sum_{j=1}^{n} c_j x_j^{(0)}$ is maximized. If in a plan period, $a_{kj}, b_k, c_j, j=1,2,\cdots,n, k=1,2,\cdots,m$ do not change, The result stays the same. That is to say, in this period, the optimal solutions of the product $X_j$ is the quantity $x_j^{(0)}, j=1,2,\cdots,n$.

In a perfect competitive market, the price $c_j$ in a plan period may fluctuate a little. We can fuzzify to $\widetilde{c_j}$. In this plan period $T$, the grade of membership of $c_j$ is not necessarily equal to 1. We let the grade of membership of $c_j$ lie in the interval $[\lambda,1], 0<\lambda<1$, (see Fig.3). Set $\widetilde{c_j}$ to be level ($\lambda$,1) i-v fuzzy number, $0<\lambda<1$.

$$\widetilde{c_j}=[(c_j-\delta_{j2},\ c_j,\ c_j+\delta_{j3};\lambda),(c_j-\delta_{j1},\ c_j,\ c_j+\delta_{j4};1)],\ j=1,2,\cdots,n \tag{18}$$

where $0<\delta_{j2}<\delta_{j1}<c_j, \quad 0<\delta_{j3}<\delta_{j4},\ j=1,2,\cdots,n$.

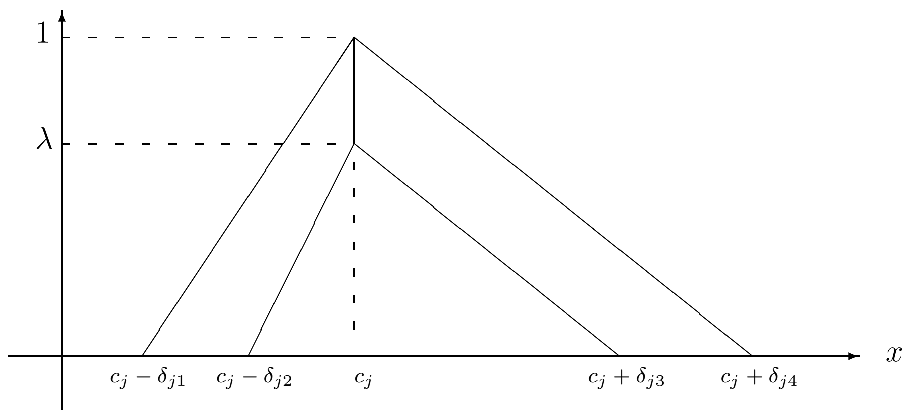


Fig.3 level $(\lambda,1)$ i-v fuzzy numbers $\widetilde{c_j}$

**3.2 Fuzzy objective function**

We denote $(x_1\widetilde{c_1})\bigoplus(x_2\widetilde{c_2})\bigoplus\cdots\bigoplus(x_n\widetilde{c_n})$ as $\sum_{j=1}^{n}\widetilde{c_j}x_j$. In eqs.(15)~(17), if we fuzzify $c_j,\ j=1,2,\cdots,n$ to level $(\lambda,1)$ i-v fuzzy numbers in a crisp linear programming, we will have the following result.

**Theorem 1** In crisp linear programming eqs.(15)~(17), we fuzzify $c_j,\ j=1,2,\cdots,n$ to eq.(18), then we have

(a) Fuzzy programming

$$\text{Maximize}\qquad \widetilde{Z}=\sum_{j=1}^{n}\widetilde{c_j}x_j \ \ \text{(by definition 9)} \tag{19}$$

subject to:

$$\sum_{j=1}^{n}a_{kj}x_j\leqq b_k,\ k=1,2,\cdots,m \tag{20}$$

$$x_j\geqq 0,\ j=1,2,\cdots,n \tag{21}$$

(b) Corresponding to (a), by definition 7, 8, 9 we get linear programming in the fuzzy sense as following:

Maximize $Z^*=\frac{1}{2}d(\widetilde{Z},\widetilde{0})$

$$=\sum_{j=1}^{n}c_jx_j+\frac{1}{16}\sum_{j=1}^{n}[\delta_{j3}-\delta_{j2}+(4-3\lambda)(\delta_{j4}-\delta_{j1})]x_j \tag{22}$$

subject to:

$$\sum_{j=1}^{n}a_{kj}x_j\leqq b_k,\ k=1,2,\cdots,m \tag{23}$$

$$x_j \geqq 0, \quad j = 1, 2, \cdots, n \tag{24}$$

Proof: (a) It follows from eqs.(15),(18) and definition 9.

(b) Since $x_j \geqq 0, j = 1, 2, \cdots, n$, by property 1, we get

$$\widetilde{Z} = [(\sum_{j=1}^{n}(c_j - \delta_{j2})x_j, \sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n}(c_j + \delta_{j3})x_j; \lambda),$$
$$(\sum_{j=1}^{n}(c_j - \delta_{j1})x_j, \sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n}(c_j + \delta_{j4})x_j; 1)]$$

Through definition 7, we obtain

$$d(\widetilde{Z}, \widetilde{0}) = 2\sum_{j=1}^{n} c_j x_j + \frac{1}{8}\sum_{j=1}^{n}[\delta_{j3} - \delta_{j2} + (4 - 3\lambda)(\delta_{j4} - \delta_{j1})]x_j$$

Using definition 8, 9 and putting them to eq.(19), we have eq.(22). This prove (b).

**Remark 1** In eq.(22), when $\delta_{j1} = \delta_{j2} = \delta_{j3} = \delta_{j4} = 0,\ j = 1, 2, \cdots, n$, this equation reduces eq.(15), i.e., $Z^* = Z$. Therefore, we take $\frac{1}{2}d(\widetilde{Z}, \widetilde{0})$ in eq.(22).

**Remark 2** In theorem 1(b), eqs.(22)$\sim$(24), the linear programming in the fuzzy sense can be found by the simplex method (or using computer package) to find the optimal solution.

**3.3 fuzzy constraints**

Suppose the sale price $c_j,\ j = 1, 2, \cdots, n$ do not vary in the plan period $T$. Similarly to §3.1, §3.2, we consider constraints of eq.(16) $\sum_{j=1}^{n} a_{kj}x_j \leqq b_k,\ k = 1, 2, \cdots, m$. We fuzzify both $a_{kj},\ b_k,\ j = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m$ as the following interval-valued fuzzy numbers, $0 < \lambda < 1$

$$\widetilde{a}_{kj} = [(a_{kj} - \delta_{kj2},\ a_{kj},\ a_{kj} + \delta_{kj3}; \lambda),\ (a_{kj} - \delta_{kj1},\ a_{kj},\ a_{kj} + \delta_{kj4}; 1)] \tag{25}$$

where $0 < \delta_{kj2} < \delta_{kj1} < a_{kj},\ 0 < \delta_{kj3} < \delta_{kj4} \quad \forall j, k$

$$\widetilde{b}_k = [(b_k - \omega_{k2},\ b_k,\ b_k + \omega_{k3}; \lambda),\ (b_k - \omega_{k1},\ b_k, , b_k + \omega_{k4}; 1)] \tag{26}$$

where $0 < \omega_{k2} < \omega_{k1} < b_k, \quad 0 < \omega_{k3} < \omega_{k4}, \quad k = 1, 2, \cdots, m.$

**Theorem 2** In eqs.(15)$\sim$(17) of the crisp linear programming, if we fuzzify $a_{kj},\ b_k,\ k = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$ to level ($\lambda$,1) i-v fuzzy numbers eqs.(25) and (26) then we have the following:

(a) Fuzzy programming

$$\text{Maximize} \qquad Z = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} \widetilde{a}_{kj} x_j \precsim \widetilde{b}_k, \; k = 1, 2, \cdots, m \tag{27}$$

$$x_j \geqq 0, \; j = 1, 2, \cdots, n \tag{28}$$

(b) Corresponding to (a), by definition 7, 8, 9 we get the linear programming in the fuzzy sense as

$$\text{Maximize} \qquad Z = \sum_{j=1}^{n} c_j x_j \tag{29}$$

subject to:

$$\sum_{j=1}^{n} a_{kj} x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{kj3} - \delta_{kj2} + (4 - 3\lambda)(\delta_{kj4} - \delta_{kj1})] x_j$$

$$\leqq b_k + \frac{1}{16} [\omega_{k3} - \omega_{k2} + (4 - 3\lambda)(\omega_{k4} - \omega_{k1})], \; k = 1, 2, \cdots, m \tag{30}$$

$$x_j \geqq 0, \; j = 1, 2, \cdots, n \tag{31}$$

Proof: (b) Using definition 8 and putting into eq.(27), we have $d(\sum_{j=1}^{n} \widetilde{a}_{kj} x_j, \widetilde{0}) \leqq d(\widetilde{b}_k, \widetilde{0}), \; k = 1, 2, \cdots, m$. From definition 7, we get eq.(30).

**3.4 Fuzzy objective function and Fuzzy constraints**

Combining §3.2 and §3.3, we have the following result.

**Theorem 3** In eqs.(15)∼(17) of the crisp linear programming, if we fuzzify $c_j, \; a_{kj}, \; b_k, \; j = 1, 2, \cdots, n, \; k = 1, 2, \cdots, m$, to level ($\lambda$,1) i-v fuzzy numbers eqs.(18)(25)(26), then we obtain

(a) Fuzzy programming

$$\text{Maximize} \qquad \widetilde{Z} = \sum_{j=1}^{n} \widetilde{c}_j x_j \tag{32}$$

subject to:

$$\sum_{j=1}^{n} \widetilde{a}_{kj} x_j \precsim \widetilde{b}_k, \; k = 1, 2, \cdots, m \tag{33}$$

$$x_j \geqq 0, \; j = 1, 2, \cdots, n \tag{34}$$

(b) Corresponding to (a), by definition 7, 8, 9 we get the linear programming in the fuzzy sense as

$$\text{Maximize} \qquad Z^* = \sum_{j=1}^{n} c_j x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{j3} - \delta_{j2} + (4 - 3\lambda)(\delta_{j4} - \delta_{j1})] x_j \tag{35}$$

subject to:

$$\sum_{j=1}^{n} a_{kj}x_j + \frac{1}{16}\sum_{j=1}^{n}[\delta_{kj3} - \delta_{kj2} + (4-3\lambda)(\delta_{kj4} - \delta_{kj1})]x_j$$
$$\leqq b_k + \frac{1}{16}[\omega_{k3} - \omega_{k2} + (4-3\lambda)(\omega_{k4} - \omega_{k1})],\ k = 1, 2, \cdots, m \tag{36}$$

$$x_j \geqq 0,\ j = 1, 2, \cdots, n \tag{37}$$

# 4 Examples

A factory produces automobils and truck. Each requires three processes. The production condition are given in table 1.

Table 1 production condition

| type | process1 hour | process2 hour | process3 hour | profit hundred dollars |
|---|---|---|---|---|
| automobil | 15 | 24 | 21 | 25 |
| truck | 30 | 6 | 14 | 48 |
| total hour | 45000 | 24000 | 28000 | |

Let the quantity of automobils and truck produced be $x_1$ and $x_2$. Then we have the following crisp linear programming

$$\text{Maximize} \qquad Z = 25x_1 + 48x_2 \ \ \text{(hundred dollars)} \tag{38}$$

subject to:

$$\begin{aligned} 15x_1 + 30x_2 &\leqq 45000 \\ 24x_1 + \ 6x_2 &\leqq 24000 \\ 21x_1 + 14x_2 &\leqq 28000 \end{aligned} \tag{39}$$

$$x_1 \geqq 0, \quad x_2 \geqq 0 \tag{40}$$

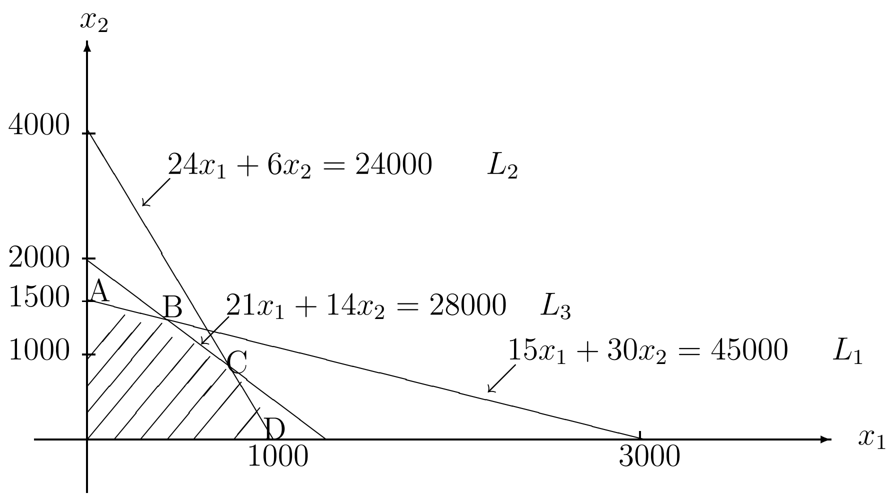


Fig.4 closed bounded convex set of (39),(40)

In this figure, $A(0, 1500)$, $B(500, 1250)$, $C(800, 800)$ and $D(1000, 0)$.

$[A]$ **Crisp case**

From Fig.4, the optimal solution of the crisp linear programming eqs.(38)∼(40) are among the points A,B,C,D. Therefore, we get $x_1 = 500(\equiv x_1^{(0)})$ and $x_2 = 1250(\equiv x_2^{(0)})$ which will maximize the profit $Z = 72500(\equiv z^{(0)})$.

$[B]$ **Fuzzy case**

Case 2.1. Let $\delta_{11} = 7,\ \delta_{12} = 6,\ \delta_{13} = 8,\ \delta_{14} = 9,\ \delta_{21} = 5,\ \delta_{22} = 4,\ \delta_{23} = 6,\ \delta_{24} = 8,\ \lambda = 0.9$.

(B.1) From theorem 1(b).

$$\text{Maximize} \qquad Z^* = 25x_1 + 48x_2 + \frac{1}{16}[4.6x_1 + 5.9x_2]$$

subject to:

$$\begin{aligned} 15x_1 + 30x_2 &\leqq 45000 \\ 24x_1 + \ 6x_2 &\leqq 24000 \\ 21x_1 + 14x_2 &\leqq 28000 \\ x_j \geqq 0, \qquad & j = 1, 2 \end{aligned}$$

Since the constraints are the same as the crisp case, from Fig.4, we need only to consider points A,B,C,D where $Z^*$ is the maximum. We have $x_1 = 500(\equiv x_1^{(1)}), x_2 = 1250(\equiv x_2^{(1)})$ and the maximum profit $Z^* = 73104.687$ (hundred dollars).

Case 2.2. Let

$$
\begin{array}{llll}
\delta_{111}=5, & \delta_{112}=1, & \delta_{113}=2, & \delta_{114}=3 \\
\delta_{121}=7, & \delta_{122}=5, & \delta_{123}=4, & \delta_{124}=8 \\
\delta_{211}=4, & \delta_{212}=3, & \delta_{213}=2, & \delta_{214}=9 \\
\delta_{221}=4, & \delta_{222}=2, & \delta_{223}=2, & \delta_{224}=5 \\
\delta_{311}=5, & \delta_{312}=4, & \delta_{313}=1, & \delta_{314}=5 \\
\delta_{321}=6, & \delta_{322}=2, & \delta_{323}=5, & \delta_{324}=8 \\
\omega_{11}=30, & \omega_{12}=20, & \omega_{13}=30, & \omega_{14}=70 \\
\omega_{21}=60, & \omega_{22}=20, & \omega_{23}=50, & \omega_{24}=60 \\
\omega_{31}=50, & \omega_{32}=10, & \omega_{33}=30, & \omega_{34}=40
\end{array}
$$

(B.2) From theorem 2(b).

$$\text{Maximize} \qquad Z = 25x_1 + 48x_2 \tag{41}$$

subject to:

$$
\begin{aligned}
15x_1 + 30x_2 + \frac{1}{16}(-1.6x_1 + 0.3x_2) &\leqq 45000 + \frac{1}{16}(62) \\
24x_1 + 6x_2 + \frac{1}{16}(5.5x_1 + 1.3x_2) &\leqq 24000 + \frac{1}{16}(30) \qquad (42) \\
21x_1 + 14x_2 + \frac{1}{16}(-3x_1 + 5.6x_2) &\leqq 28000 + \frac{1}{16}(7)
\end{aligned}
$$

$$x_1 \geqq 0, \quad x_2 \geqq 0 \tag{43}$$

From eq.(42) and (43), the closed bounded convex set L is the following:

$$
\begin{aligned}
14.9x_1 + 30.01875x_2 &\leqq 45003.875 \\
24.34375x_1 + 6.08125x_2 &\leqq 24001.875 \qquad (44) \\
20.8125x_1 + 14.35x_2 &\leqq 28000.4375
\end{aligned}
$$

$$x_1 \geqq 0, \quad x_2 \geqq 0 \tag{45}$$

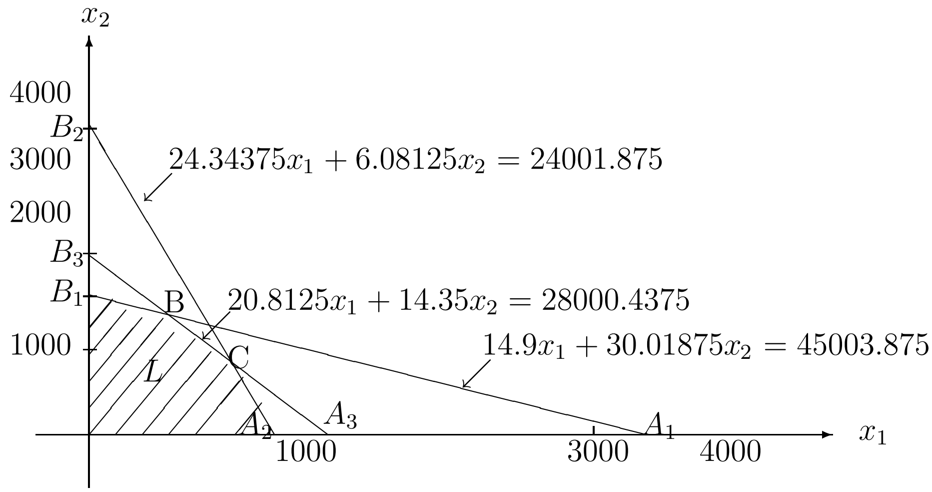


Fig.5 closed bounded convex set of (44),(45)

In Fig. 5, $A_1$(3020.394,0), $B_1$(0,1499.192), $A_2$(985.956,0), $B_2$(0,3946.865), $A_3$(1345.366,0), $B_3$(0,1951.25). The vertices of L are $B_1$, B(475.353,1261.822) ,C(781.756,817.432), $A_2$(985.956,0). Since the optimal solution must be integers, we consider the points in $L$ which are closest to point $B_1$, B, C, $A_2$. Here we take points $B_1^*$(0,1499), $B^*$(475,126), $C^*$(781,817) and $A_2$(985,0). The optimal solution of eq.(41) occurs when $x_1 = 475(\equiv x_1^{(2)}),\ x_2 = 1261(\equiv x_2^{(2)})$ and the maximum profit is $Z = 72403$.

# 5 Discussion

(A) The crisp case is a special case of the fuzzy case.

(a) In theorem 1(b), let $\delta_{j2} = \delta_{j3}$ and $\delta_{j1} = \delta_{j4},\ j = 1, 2, \cdots, n$. Then, in theorem 1(b), eqs.(22)∼(24) reduces to

$$\text{Maximize} \quad Z^* = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{kj} x_j \quad \leqq b_k, \quad k = 1, 2, \cdots, m$$
$$x_j \geqq 0, \quad j = 1, 2, \cdots, n$$

this is the crisp case of eqs.(15)∼(17). Therefore, the crisp case of eqs.(15)∼(17) is a special case of theorem 1(b).

(b) In theorem 2(b), let $\delta_{kj2} = \delta_{kj3}$ and $\delta_{kj1} = \delta_{kj4},\ j = 1, 2, \cdots, n$ ,$\omega_{k2} = \omega_{k3}$ and $\omega_{k1} = \omega_{k4}$ for all $k = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$, then in theorem 2(b), eqs.(29)$\sim$(31) reduce to the crisp case of eqs.(15)$\sim$(17). Therefore, the crisp case of eqs.(15)$\sim$(17) is a special case of theorem 2(b).

(c) In theorem 1,2(b), each are special case of theorem 3(b).

(c1) In theorem 3(b), let $\delta_{kj2} = \delta_{kj3}$ and $\delta_{kj1} = \delta_{kj4}$ , $\omega_{k2} = \omega_{k3}$ and $\omega_{k1} = \omega_{k4}$ for all $k = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$,
then theorem 3(b) eqs.(35)$\sim$(37) reduce to theorem 1(b) eqs.(22)$\sim$(24). Therefore, theorem 1(b) is a special case of theorem 3(b).

(c2) In theorem 3(b), let $\delta_{j2} = \delta_{j3}$ and $\delta_{j1} = \delta_{j4},\ j = 1, 2, \cdots, n$, then theorem 3(b) eqs.(35)$\sim$(37) reduce to theorem 2(b) eqs.(29)$\sim$(31). Therefore, theorem 2(b) is a special case of theorem 3(b).

(B) The result of fuzzification by fuzzy numbers is a special case of fuzzification by interval-valued fuzzy numbers.

(b1) In theorem 1 eq.(18), let $\delta_{j3} = \delta_{j2} = 0$ for all j and $\lambda = 0$. From Fig.3, we have level ($\lambda$,1) i-v fuzzy number in eq.(18) reduce to fuzzy number $\widetilde{c_j} = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1),\ j = 1, 2, \cdots, n$. This implies eq.(19) in theorem 1(a) use fuzzy numbers $\widetilde{c_j} = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1),\ j = 1, 2, \cdots, n$.
In theorem 1(b), eq.(22). $Z^* = \sum_{j=1}^{n} c_j x_j + \frac{1}{4}\sum_{j=1}^{n}(\delta_{j4} - \delta_{j1})x_j$ is the result of defuzzification by signed distance using fuzzy numbers $\widetilde{c_j} = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1)$, through

$$d(\widetilde{c_j}, \widetilde{0}) = \frac{1}{2}\int_0^1 (\widetilde{c}_{jL}(\alpha) + \widetilde{c}_{jU}(\alpha))d\alpha = c_j + \frac{1}{4}(\delta_{j4} - \delta_{j1})$$

Therefore, the defuzzification by using fuzzy numbers is a special case of using level ($\lambda$,1) i-v fuzzy numbers.

(b2) In theorem 2(a), let $\delta_{kj2} = \delta_{kj1} = 0,\ \omega_{k2} = \omega_{k1} = 0$ for all $j, k$, and $\lambda = 0$. It is similarly to (b1). Level ($\lambda$,1) i-v fuzzy numbers in eq.(25) reduce to fuzzy numbers $\widetilde{a}_{kj} = (a_{kj} - \delta_{kj1},\ a_{kj}, a_{kj} + \delta_{kj4}; 1)$, and level ($\lambda$,1) i-v fuzzy neumbers in eq.(26) reduce to fuzzy number $\widetilde{b}_k = (b_k - \omega_{k1}, b_k, b_k + \omega_{k4}; 1)$ for all $j, k$. $\widetilde{a}_{kj}, \widetilde{b}_k, k = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ in eq.(27) of theorem 2(a) are all fuzzy numbers. Eq.(30) in theorem 2 becomes

$$\sum_{j=1}^{n} a_{kj}x_j + \frac{1}{4}\sum_{j=1}^{n}(\delta_{kj4} - \delta_{kj1})x_j \leqq b_k + \frac{1}{4}(\omega_{k4} - \omega_{k1}),\ k = 1, 2, \cdots, m$$

This is the result of defuzzification of fuzzy number through the signed distance $d(\widetilde{a}_{kj},\widetilde{0}) = a_{kj}+ \frac{1}{4}\sum_{j=1}^{n}(\delta_{kj4}-\delta_{kj1})$, $d(\widetilde{b}_k,\widetilde{0}) = b_k + \frac{1}{4}(\omega_{k4}-\omega_{k1})$. Therefore, it has the same conclusion as (b1). The defuzzification result by using fuzzy numbers is a special case of using level ($\lambda$,1) i-v fuzzy numbers.

(b3) In theorem 3, the same treatments will lead to the same conclusions as (b1) and (b2).

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