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Soliton Solutions Obtained from the Duffing Equation

M. Bhatti* and P. Bracken†

* Department of Physics and Geology
†Department of Mathematics
University of Texas Pan American
Edinburg, TX 78541-2999, USA

Abstract

The Duffing equation is studied under a large class of potential functions which have a polynomial form. It is shown that the equation can be put in the form of a quadrature and some specific cases are calculated. In particular, solutions with a typical soliton profile are found. Finally, numerical solutions are presented graphically along with the corresponding phase portrait.

The equation describing a nonlinear oscillator which was first introduced by Duffing [**1,2,3**] with a cubic stiffness to describe a hard spring has become, together with van der Pol's equation a very common example of a nonlinear oscillator. Moon and Holmes [**4**] showed that the Duffing equation in the form

$$\ddot{y}(t) + \delta \dot{y}(t) - y(t) + y^3(t) = \gamma \cos \omega t \tag{1}$$

provides the simplest possible model for the forced vibrations of a cantilever beam in the nonuniform field of two permanent magnets. There are two domains in which the attractive forces overcome the elastic forces which would keep the beam straight, and in the absence of a driving force, the beam settles with its tip close to one or other of the magnets. There is also an unstable, central equilibrium at which the magnetic forces cancel. We will consider a potential function which has some of these properties, in which case, the Duffing equation describes the motion of a classical particle in a double or single well depending on whether n takes even or odd values in the equation below

$$V(y) = -\frac{1}{2} a y^2 - \frac{1}{2} b y^n. \tag{2}$$

The force law is given by $F = -dV/dy$, and the corresponding differential equation is obtained by using Newton's second law of motion. If we let $\dot{y} = v$ and $\dot{v} + ay + \frac{1}{2}bny^{n-1} = 0$, then this gives the energy integral when the mass is set equal one of the form

$$\frac{1}{2}v^2 + V(y) = E. \tag{3}$$

Using (2), a generalization of equation (1) will be obtained with both the first derivative term and harmonic term absent. Of course E is constant on each orbit and often the orbit is a closed oval. The time traversing this orbit once will give the period.

Specific solutions of the Duffing equation which have the typical form of solitons can be obtained by direct integration. These classes of solution have not been fully considered before. Consider then the generalized Duffing's equation which we take to have the following form

$$\ddot{y}(t) + ay(t) + \frac{1}{2}bny(t)^{n-1} = 0. \tag{4}$$

In (4) a and b are real constants and n is a real parameter. In the case $n = 0$, we consider the equation to have the last term replaced by a $y(t)$ which is the equation of the simple harmonic oscillator that has known solutions. We will be primarily interested in the case in which n is a positive integer. A solution for (4) can be written in the form of a quadrature as follows. Multiplying both sides of (4) by $\dot{y}$, the equation takes the form

$$(\dot{y}(t)^2)^{\cdot} + a(y^2)^{\cdot} + b(y^n)^{\cdot} = 0. \tag{5}$$

In this form the equation can be integrated with respect to t and we obtain

$$\dot{y}^2 + ay^2 + by^n = C. \tag{6}$$

Once values for $y(0)$ and $\dot{y}(0)$ are specified, the constant C can be evaluated as

$$C = \dot{y}(0)^2 + ay(0)^2 + b(y(0))^n. \tag{7}$$

Solving (5) for $\dot{y}^2$, equation (4) can finally be reduced to the form of a quadrature

$$\int \frac{dy}{\sqrt{C - ay^2 - by^n}} = \epsilon(t + K), \quad \epsilon = \pm 1. \tag{8}$$

Here K appears as a final integration constant. The integral in (8) can be done exactly for many choices of the constants, or expressed in terms of special functions. Some consequences of the form of solution (8) will be given.

(i) The case in which $a = -1$, $b = 2$ and $C = 0$ can be worked out in closed form. Then (4) takes the form $\ddot{y} - y + ny^{n-1} = 0$ and solutions are obtained from

$$\int \frac{dy}{\sqrt{y^2 - 2y^n}} = \epsilon(t + K), \tag{9}$$

with

$$\dot{y}(0) = 0, \quad y(0) = \frac{1}{2})^{1/(n-2)}, \quad V(y) = \frac{1}{2}y^2 - y^n.$$

Completing the integral, this has the form

$$(1 - 2y(t)^{n-2})^{1/2} = \tanh(\epsilon(\frac{n-2}{2})(t + K)),$$

and therefore

$$y(t)^{n-2} = \frac{1}{2}\text{sech}^2(\epsilon(\frac{n-2}{2})(t + K)).$$

When $n \neq 2$, this can be solved for y explicitly

$$y(t) = 2^{-\frac{1}{n-2}}\text{sech}^{\frac{2}{n-2}}(\epsilon(\frac{n-2}{2})(t + K)). \tag{10}$$

It is worth noting that for the case $n = 4$, this solution reduces to the form

$$y(t) = \frac{1}{\sqrt{2}}\text{sech}(\epsilon(t + K)). \tag{11}$$

(ii) For the case in which $a = 1$, $b = -2$, $\ddot{y} + y - ny^{n-1} = 0$ and $V(y) = -1/2y^2 + y^n$. Under the initial conditions $y(0) = (1/2)^{1/n-2}$ and $\dot{y}(0) = 0$, equation (4) can be integrated to give the closed form solution

$$y(t) = (\frac{1}{2})^{\frac{1}{n-2}} \sec^{\frac{2}{n-2}}(\epsilon(\frac{n-2}{2})(t + K)) \tag{12}$$

(iii) Another class of solution arises from the case in which $C = 1$, $a = 2$ and $b = -1$ with power $n = 4$ so equation (4) takes the form $\ddot{y}(t) + 2y(t) - 2y(t)^3 = 0$. The solution can be written with $V(y) = -y^2 + \frac{1}{2}y^n$

$$y(t) = \tanh(\epsilon(t + K)). \tag{13}$$

(iv) More well known oscillatory solutions can be found, in particular, in terms of elliptic functions when $n = 4$, as is well known. Consider the specific form of the equation $\ddot{y}(t) + y(t) - ny(t)^{n-1} = 0$ with $a = 1$ and $b = -2$ in equation (4) under the conditions $y(0) = a$, $\dot{y}(0) = 0$. The following integral will define a solution in terms of a special function

$$\int \frac{dy}{\sqrt{a^2 - y^2 - (a^n - y^n)}} = \epsilon(t + K). \tag{14}$$

A particularly powerful way to investigate properties as well as solutions of (4) is to do numerical studies for particular values of the parameters. A series of plots which display solutions and contain information concerning the phase space behavior and potential functions are given.

(i) In Figure 1, a soliton solution for the equation

$$\ddot{y} - y = -ny^{n-1}, \qquad y(0) = (\frac{1}{2})^{1/(n-2)}, \quad \dot{y}(0) = 0, \tag{15}$$

for the case $n = 3$ is given. The curve has a typical soliton profile, and resembles a form of solution which arises from other nonlinear integrable systems. The phase space plot for this case is given in Figure 2. Notice that the phase portrait is a closed curve in this case. The potential function $V(y)$ is drawn for values of $n = 3$ and 4 in Figure 3.

(ii) In Figure 4, a solution for the equation

$$\ddot{y} + y = ny^{n-1}, \qquad y(0) = (\frac{1}{2})^{1/(n-2)}, \qquad \dot{y}(0) = 0, \tag{16}$$

is given for the case in which $n = 4$. The solution curve has a global minimum at $t = 0$ is decreasing for $t < 0$ and increasing for $t > 0$. The corresponding phase plot and potential functions are given in Figures 5 and 6 respectively. In this case, the phase plot does not close as in the previous example.

(iii) Next, the case of the equation

$$\ddot{y} + 2y = (n/2)y^{n-1}, \qquad y(0) = 1, \qquad \dot{y}(0) = 0, \tag{17}$$

is given for $n = 5$. In this case, the solution curve given in Figure 7 is increasing then decreasing with a local maximum at zero. The phase portrait and potential functions are given in Figures 8 and 9 as well.

(iv) The same equation that was studied in (iii) above is considered in Figure 10, but but with $n = 4$ and subjected to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$. It is interesting to note that this small change is sufficient to generate a solution which oscillates as a function of the independent variable.

The phase portrait, which is closed as it should be in such a case and curves for $V(y)$ are shown in Figures 11 and 12.

(v) The equation and initial conditions

$$\ddot{y} + y = ny^{n-1}, \qquad y(0) = 1 \qquad \dot{y}(0) = 0 \tag{18}$$

and $n = 5$ generates a solution which is similar to that of Figure 4. This is presented in Figure 13, while the phase portrait and curves for the potential $V(y)$ are given in 14 and 15, respectively.

To conclude, the explicit solutions given here are complement other methods for generating solutions such as numerical ones. We have also solved this equation using a numerical method based on a Bernstein polynomial expansion, but will not be reported here. An example of a nonlinear equation solved in this manner is given in [**5**].

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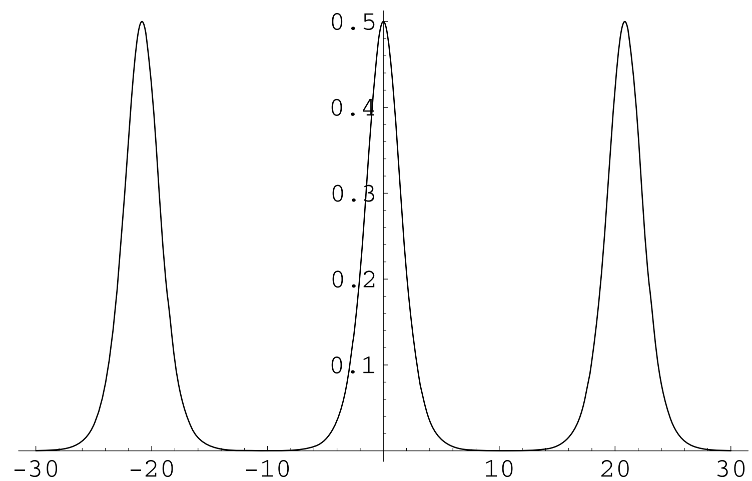

Figure 1: Plot of soliton solution for (11) with $n = 3$

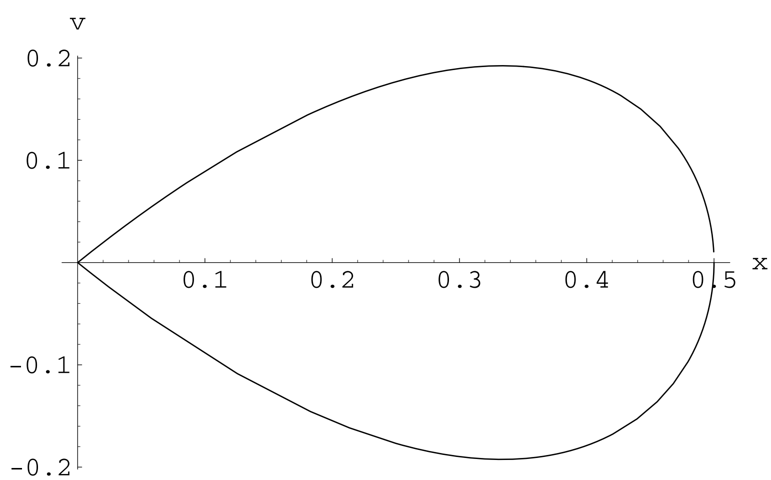

Figure 2: Phase Space Plot for (11) with $n = 3$

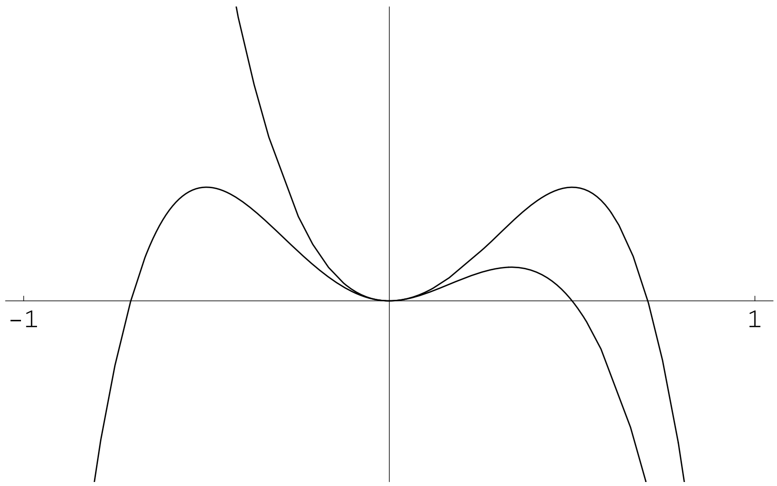

Figure 3: Plot of potential function for n even and odd

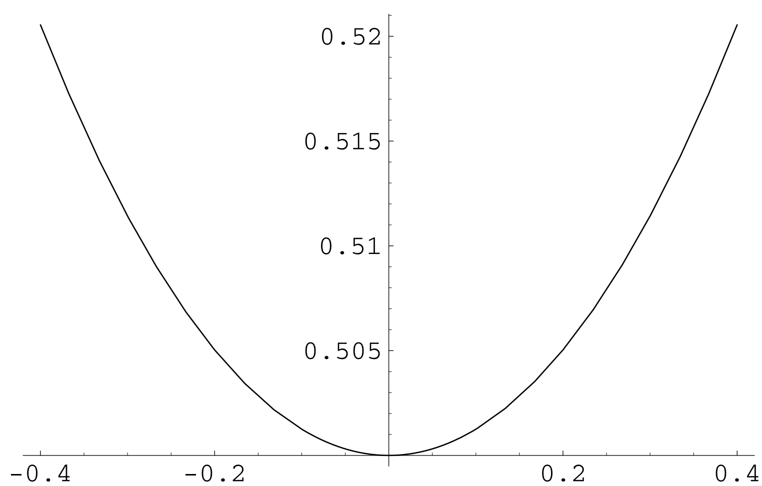

Figure 4: Solution plot for (12) when $n = 4$

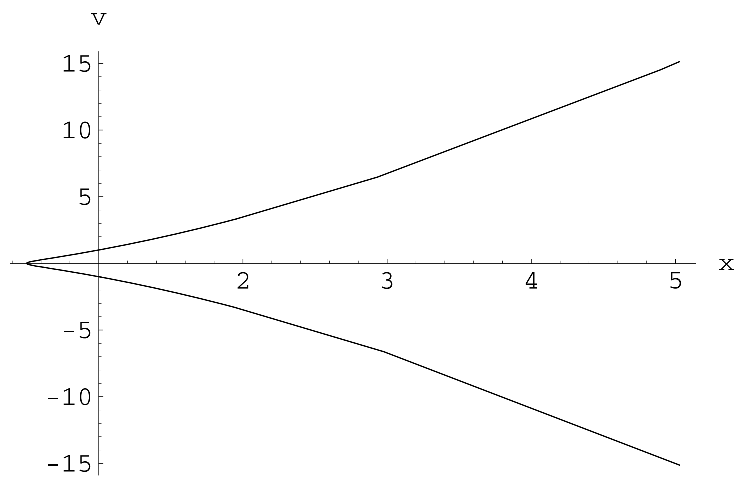

Figure 5: Phase space plot for solution in Figure 4

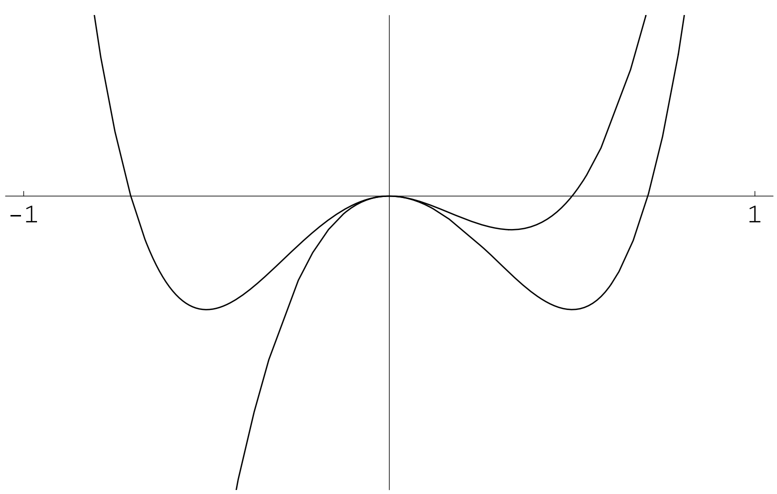

Figure 6: Plot of potential function for even and odd n

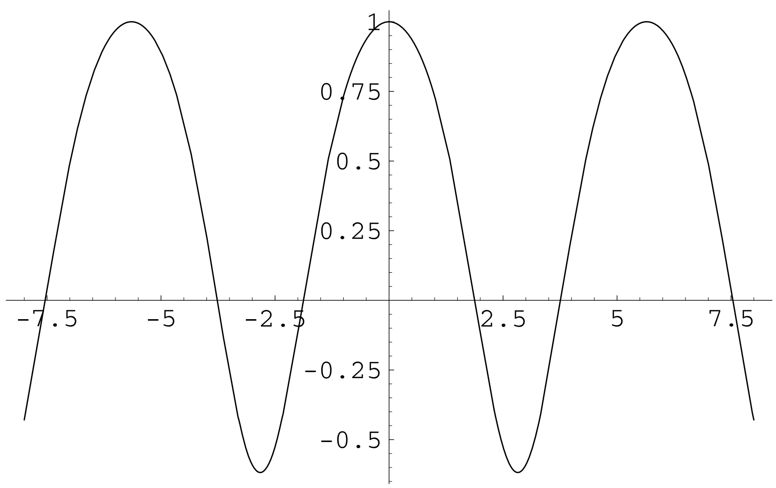

Figure 7: Plot of solution for (13) with $n = 5$

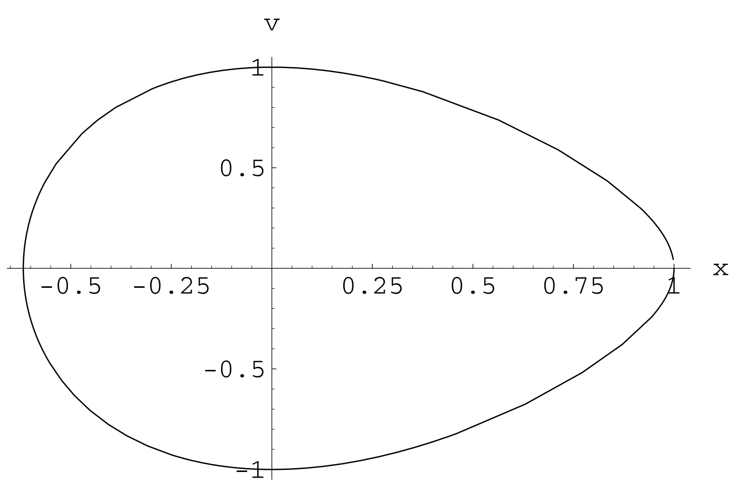

Figure 8: Phase space plot for solution in Figure 7.

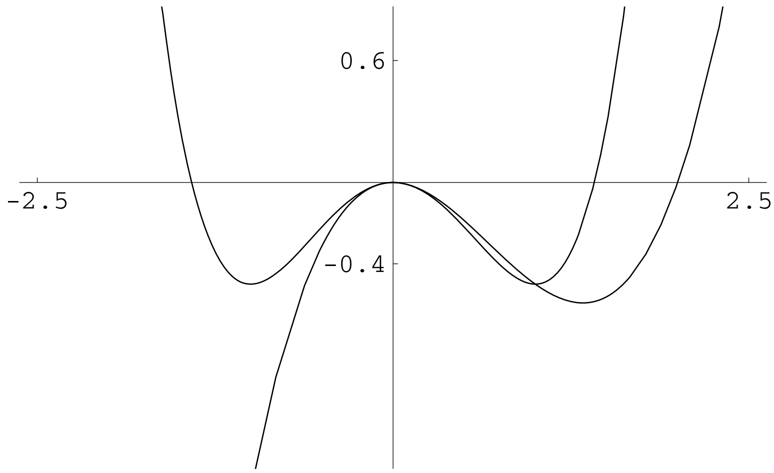

Figure 9: Plot of corresponding potential function for even and odd n values.

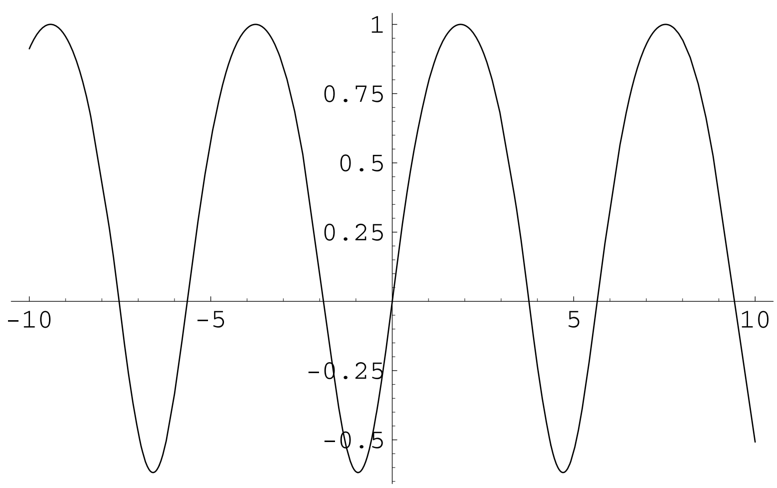

Figure 10: Plot of solution for (13) with $n = 4$

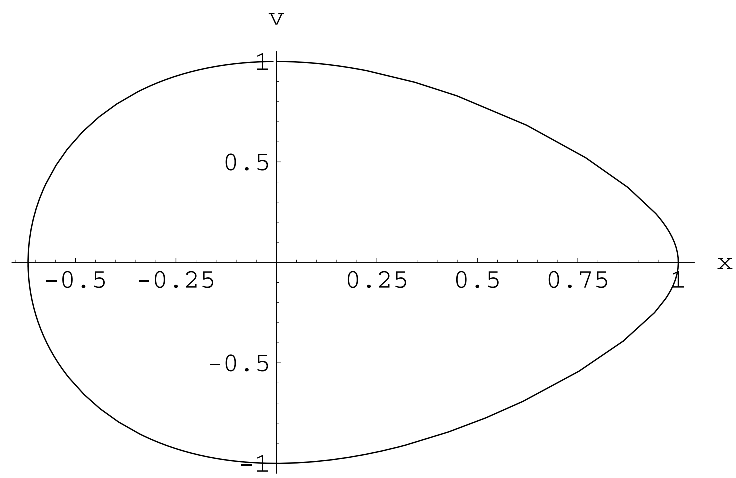

Figure 11: Phase space plot for the solution of Figure 10.

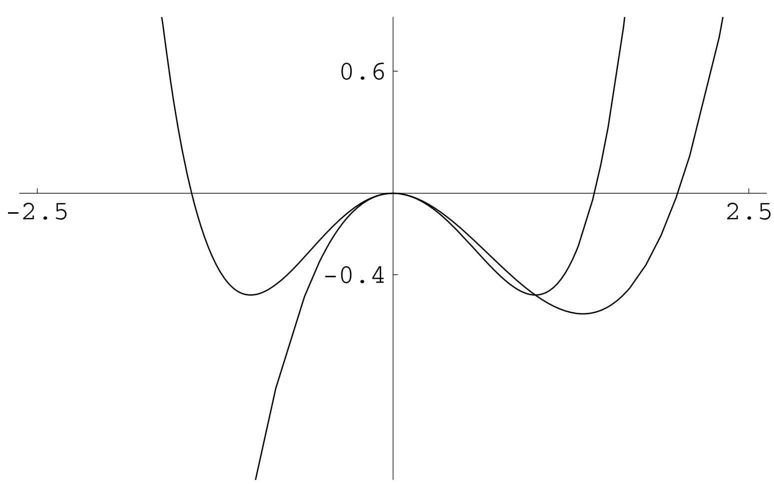

Figure 12: Plot of potential function for Figure 10.

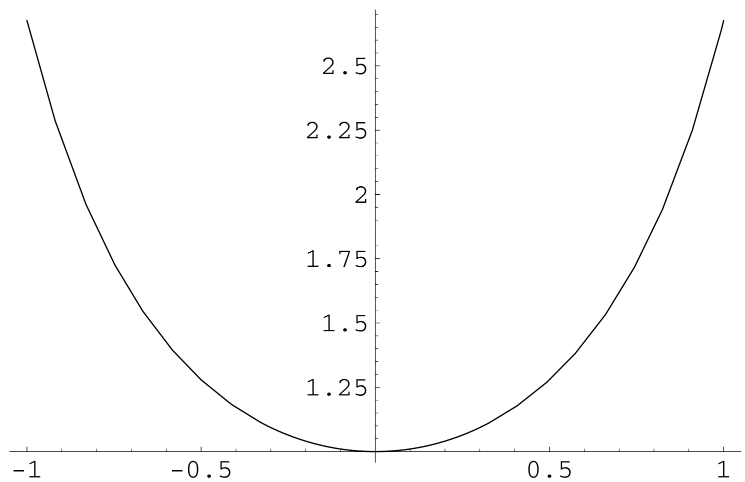

Figure 13: Plot of solution for equation (14) with $n = 5$

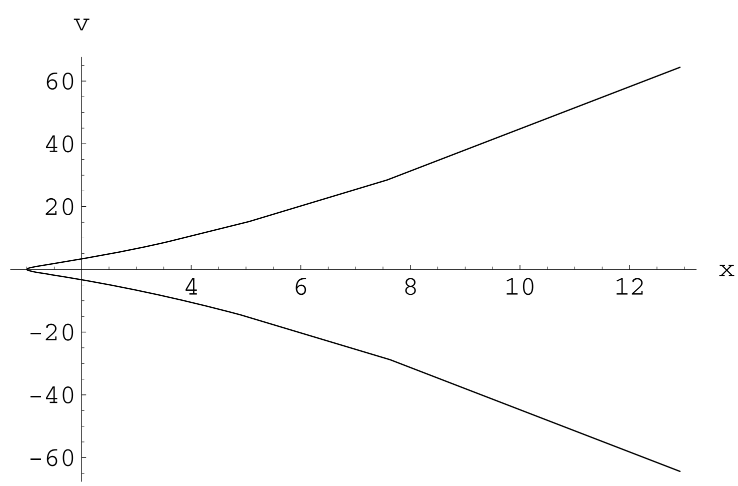

Figure 14: Phase space plot for solution in Figure 13.

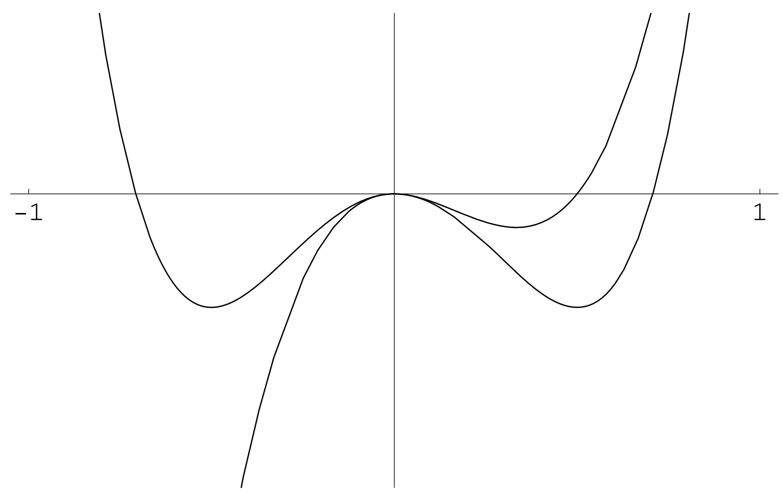

Figure 15: Plot of potential function for even and odd n values.