Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 30, 1487 - 1492

On Jordan and Jordan*-Generalized Derivations in Semiprime Rings with Involution

M. N. Daif

Department of Mathematics, Faculty of Science
Al-Azhar University, Nasr City (11884), Cairo, Egypt
nagydaif@yahoo.com

M. S. Tammam El-Sayiad

Department of Mathematics, Faculty of Science
Beni Suef University, Beni Suef (62111), Egypt
m_s_tammam@yahoo.com

Abstract

The purpose of this note is to prove the following result. Let R be a $6-$torsion free semiprime *-ring and let G: $R \longrightarrow R$ be an additive mapping such that $G(xyx) = G(x)y^*x^* + xD(y)x^* + xyD(x)$ holds for all $x, y \in R$ and some *-derivations D of R. Then G is a Jordan*-generalized derivation.

Mathematics Subject Classification: 16W10, 39B05

Keywords: ring, *-ring, prime ring, semiprime ring, derivation, Jordan derivation, generalized derivation, Jordan generalized derivation, *-derivation, Jordan *-derivation, *-generalized derivation, Jordan *-generalized derivation

1 Introduction

This note is motivated by the work of Vukman [9]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, if $nx = 0$, $x \in R$ implies $x = 0$, where n is a positive integer. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $x \longrightarrow x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ for all pairs $x, y \in R$. An additive mapping $D : R \rightarrow R$, where R is a *-ring, is called a *-derivation in case

$D(xy) = D(x)y^* + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan*-derivation if $D(x^2) = D(x)x^* + xD(x)$ holds for all $x \in R$. The concepts of *-derivation and Jordan*-derivation were first mentioned in [4]. An additive mapping $T : R \longrightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. An additive mapping $D : R \longrightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner if there exist $a \in R$ such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on 2-torsion free prime ring is a derivation. Cusack [5] generalized Herstein's theorem to 2-torsion free semiprime rings.

In [6], Hvala has defined the notion of generalized derivation as follows: An additive mapping $G : R \to R$ is said to be a generalized derivation if there exists a derivation $D : R \to R$ such that

$$G(xy) = G(x)y + xD(y) \text{ for all } x, y \in R.$$

Also, he called the maps of the form $x \to ax + xb$ where a, b are fixed elements in R by the inner generalized derivations. By a similar fashion to the definition of the *-derivation and the Jordan *-derivation we can define the concepts of a *-generalized derivation and a Jordan *-generalized derivation as follows, an additive mapping $G : R \to R$ is said to be a *-generalized derivation if there exists a *-derivation $D : R \to R$ such that

$$G(xy) = G(x)y^* + xD(y) \text{ for all } x, y \in R.$$

And an additive mapping $G : R \to R$ is said to be a Jordan *-generalized derivation if there exists a *-derivation $D : R \to R$ such that

$$G(x^2) = G(x)x^* + xD(x) \text{ for all } x \in R.$$

Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in R$) and the concept of a *-generalized derivation covers both the concepts of a *-derivation and a left *-centralizer (i.e., an additive map f satisfying $f(xy) = f(x)y^*$ for all $x, y \in R$). In [1, Remark 1] Brešar proved that: for a semiprime ring R, if G is a function from R to R and $D : R \to R$ is an additive mapping such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$, then D is uniquely determined by G and moreover G must be a derivation. Ashraf and Nadeem-Ur-Rehman, In [8], proved the following result.

Theorem 1.1 ([8], Theorem PP. 7). *Let R be a 2-torsion free ring such that R has a commutator which is not a zero divisor. Then every Jordan generalized derivation on R is a generalized derivation.*

An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. Of course any derivation is a Jordan triple derivation. In [3], Brešar and Vukman proved that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation. In [2], Brešar proved that if R is a 2-torsion free semiprime ring and $D : R \to R$ is a Jordan triple derivation, then D is a derivation. In [9], Vukman proved the following theorem.

Theorem 1.2 ([9] Theorem 1). *Let R be a 6-torsion free semiprime *-ring and let D: $R \longrightarrow R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x), \tag{1}$$

*for all $x, y \in R$. Then D is a Jordan *-derivation.*

2 The Main Result

In this note we give an answer to to Vukmun's theorem in case of the Jordan *-generalized derivation.

Theorem 2.1. *Let R be a 6-torsion free semiprime *-ring and let G: $R \longrightarrow R$ be an additive mapping satisfying the relation*

$$G(xyx) = G(x)y^*x^* + xD(y)x^* + xyD(x),$$

*for all $x, y \in R$ and some Jordan *-derivations D of R. Then G is a Jordan *-generalized derivation.*

Proof. We have the assumption

$$G(xyx) = G(x)y^*x^* + xD(y)x^* + xyD(x), \quad x, y \in R. \tag{2}$$

Replacing y by xyx in (2) we get

$$G(x^2yx^2) = G(x)x^*y^*x^{*2} + xD(xyx)x^* + x^2yxD(x), \quad x, y \in R. \tag{3}$$

Using (1) in (3) we obtain

$$\begin{aligned} G(x^2yx^2) = G(x)x^*y^*x^{*2} + xD(x)y^*x^{*2} + x^2D(y)x^{*2} \\ +x^2yD(x)x^* + x^2yxD(x), \quad x, y \in R. \end{aligned} \tag{4}$$

On the other hand replacing x by x^2 in (2) we get

$$G(x^2yx^2) = G(x^2)y^*x^{*2} + x^2D(y)x^{*2} + x^2yD(x^2), \quad x, y \in R. \tag{5}$$

Since D is a Jordan *-derivation, (5) may be rewritten as

$$G(x^2yx^2) = G(x^2)y^*x^{*2} + x^2D(y)x^{*2} + x^2yD(x)x^* + x^2yxD(x), \quad x, y \in R. \tag{6}$$

Subtracting (4) from (6) we obtain

$$A(x)y^*x^{*2} = 0, \quad x, y \in R, \tag{7}$$

where $A(x)$ stands for $G(x^2) - G(x)x^* - xD(x)$. We intend to prove that

$$A(x) = 0, \quad x \in R. \tag{8}$$

Replacing y by y^* in (7) we obtain

$$A(x)yx^{*2} = 0, \quad x, y \in R. \tag{9}$$

Right multiplication by $A(x)$ and left multiplication by x^{*2} in (9), we have $x^{*2}A(x)yx^{*2}A(x) = 0$, for all $x, y \in R$. Since R is semiprime we obtain

$$x^{*2}A(x) = 0, \quad x \in R. \tag{10}$$

Replacing y by $x^{*2}yA(x)$ in (9), and by the semiprimeness of R we get

$$A(x)x^{*2} = 0, \quad x \in R. \tag{11}$$

A linearization of (11) gives

$$\begin{aligned} A(x)y^{*2} + A(y)x^{*2} + B(x, y)x^{*2} + B(x, y)y^{*2} + A(x)(xy + yx)^* \\ +A(y)(xy + yx)^* + B(x, y)(xy + yx)^* = 0, \quad x, y \in R, \end{aligned} \tag{12}$$

where $B(x, y)$ stands for $G(xy + yx) - D(x)y^* - D(y)x^* - xD(y) - yD(x)$. Putting $-x$ for x in the above relation and comparing the relation so obtained with the relation (12) we obtain

$$B(x, y)x^{*2} + B(x, y)y^{*2} + A(x)(xy + yx)^* + A(y)(xy + yx)^* = 0, \quad x, y \in R. \tag{13}$$

The substitution $2x$ for x in (13) gives

$$4B(x, y)x^{*2} + B(x, y)y^{*2} + 4A(x)(xy + yx)^* + A(y)(xy + yx)^* = 0, \quad x, y \in R. \tag{14}$$

Subtracting the relation (13) from (14) we obtain $3B(x, y)x^{*2} + 3A(x)(xy + yx)^* = 0$, $x, y \in R$ which gives

$$B(x, y)x^{*2} + A(x)(xy + yx)^* = 0, \quad x, y \in R. \tag{15}$$

Right multiplication of the above relation by $A(x)x^*$ and using (10) we get

$$A(x)y^*x^*A(x)x^* + A(x)x^*y^*A(x)x^* = 0, \quad x, y \in R. \tag{16}$$

Subtracting in (16) yx for y and multiplying (16) from the left side by x^* we obtain $x^*A(x)x^*y^*x^*A(x)x^* = 0, \quad x, y \in R$. Replacing y by y^* so by the semiprimeness of R we obtain $x^*A(x)x^* = 0$, $x \in R$. Now the relation (16) reduces to $(A(x)x^*)y^*(A(x)x^*) = 0$, $x, y \in R$, which gives

$$A(x)x^* = 0, \quad x \in R. \tag{17}$$

Now the relation (15) reduces to $B(x, y)x^{*2} + A(x)y^*x^* = 0$, $x, y \in R$. Right multiplication of this relation by $A(x)$ and left multiplication by x^* and replacing y by y^* gives $(x^*A(x))y(x^*A(x)) = 0$, $x, y \in R$, which gives

$$x^*A(x) = 0, \quad x \in R. \tag{18}$$

Linearizing relation (17) gives

$$A(x)y^* + A(y)x^* + B(x, y)x^* + B(x, y)y^* = 0, \quad x, y \in R. \tag{19}$$

Putting $-x$ for x in the above relation and comparing the relation so obtained with the relation (19) we obtain

$$A(x)y^* + B(x, y)x^* = 0, \quad x, y \in R. \tag{20}$$

Right multiplication of the above relation by $A(x)$ gives using (18) that $A(x)y^*A(x) = 0$, $x, y \in R$, which yields $A(x) = 0$, for all $x \in R$. In other words $G(x^2) = G(x)x^* + xD(x)$, for all $x \in R$ which means that G is a Jordan *-generalized derivation.

It is clear that if we use the *-derivation D to be the zero *-derivation, in the above theorem, we get

Corollary 2.2. *Let R be a 6-torsion free semiprime *-ring and let $T\colon R \longrightarrow R$ be an additive mapping. If $T(xyx) = T(x)y^*x^*$ for all $x, y \in R$, then T is a left Jordan *-centralizer.*

References

[1] M. Brešar, On the distance of the compositions of two derivations to generalized derivations, Glasgow Math. J. **33** (1991), 80-93.

[2] M. Brešar, Jordan mappings of semiprime rings, Journal of Algebra, **127** (1989), 218-228.

[3] M. Brešar and J. Vukman, Jordan derivations on prime rings, Bull. Austral. Math. Soc. **37** (1988), 321-322.

[4] M. Brešar and J. Vukman, On some additive mappings in rings with involution, Aequationes Math. **38**(1989), 178-185.

[5] J. Cusack, Jordan derivations on semiprime rings, Proc. Amer. Math. Soc. **53** (1975), 321-324.

[6] B. Hvala, Generalized derivations in rings, Comm. Alg. **26**(4) (1998), 1147-1166.

[7] I. N. Herstein; Jordan derivations of prime rings, Proc. Amer. Math. Soc. **8** (1957), 1104-1119.

[8] Mohammad Aahraf and Nadeem-Ul-Rehman, On Jordan generalized derivations in rings, Math. J. Okayama Univ. **42**(2000), 7-9.

[9] J. Vukman, A note on Jordan*- derivations in semiprime rings with involution , International Mathematical Forum, **1**, 2006, no. 13, 617-622.

Received: June 16, 2007