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On Generalized Derivations in Semiprime Rings

M. N. Daif

Department of Mathematics, Faculty of Science
Al-Azhar University, Nasr City (11884), Cairo, Egypt
nagydaif@yahoo.com

M. S. Tammam El-Sayiad

Department of Mathematics, Faculty of Science
Beni Suef University, Beni Suef (62111), Egypt
m_s_tammam@yahoo.com

Abstract

The purpose of this note is to prove the following result. Let R be a semiprime ring of characteristic not 2 and G: $R \longrightarrow R$ be an additive mapping such that $G(x^2) = G(x)x + xD(x)$ holds for all $x \in R$ and some derivations D of R. Then G is generalized derivation.

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1 Introduction

This note is motivated by the work of Zalar [6]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, if $nx = 0$, $x \in R$ implies $x = 0$, where n is a positive integer. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $T : R \longrightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in R$ and is called a Jordan left (right) centralizer in case $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$ holds for all $x \in R$. A result of Zalar [6] asserts that any Jordan centralizer on a semiprime ring of characteristic not 2 is a centralizer. An additive mapping $D : R \longrightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$

and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [4] asserts that any Jordan derivation on 2-torsion free prime ring is a derivation. Cusack [2] generalized Herstein's theorem to 2-torsion free semiprime ring.

In [3], Hvala has defined the notion of a generalized derivation as follows: An additive mapping $G : R \to R$ is said to be a generalized derivation if there exists a derivation $D : R \to R$ such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$. Also, he called the maps of the form $x \to ax + xb$ where a, b are fixed elements in R by the inner generalized derivations. Ashraf and Nadeem-Ur-Rehman, in [5], have defined the concept of a Jordan generalized derivation as follows: An additive mapping $G : R \to R$ is said to be a Jordan generalized derivation if there exists a derivation $D : R \to R$ such that $G(x^2) = G(x)x + xD(x)$ for all $x \in R$. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizers and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizers. In [1, Remark 1] Brešar proved that for a semiprime ring R, if G is a function from R to R and $D : R \to R$ is an additive mapping such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$, then D is uniquely determined by G and moreover G must be a derivation. Ashraf and Nadeem-Ur-Rehman, in [5], proved the following result: Let R be a 2-torsion free ring such that R has a commutator which is not a zero divisor, then every Jordan generalized derivation on R is a generalized derivation.

In this note, using Zalar's method, we study the same result of Ashraf and Nadeem-Ur-Rehman but for a semiprime ring, and without the condition of zero divisor, i.e., if R is a semiprime ring of characteristic not 2 and G is an additive mapping which satisfies

$$G(x^2) = G(x)x + xD(x)$$

holds for all $x \in R$ and some derivation D of R, then G is generalized derivation. This result will be a generalization of the result of Zalar [6]. In order to prove our result we will need the following lemmas which are due to Zalar.

Lemma 1.1 ([6] Lemma 1.1). *Let R be a semiprime ring. If $a, b \in R$ are such that $axb = 0$ for all $x \in R$, then $ab = ba = 0$.*

Lemma 1.2 ([6] Lemma 1.2). *Let R be a semiprime ring and $\theta, \phi : R \times R \to R$ biadditive mappings. If $\theta(x, y)w\phi(x, y) = 0$ for all $x, y, w \in R$, then $\theta(x, y)w\phi(u, v) = 0$ for all $x, y, u, v, w \in R$.*

Lemma 1.3 ([6] Lemma 1.3). *Let R be a semiprime ring and $a \in R$ be some fixed element. If $a[x, y] = 0$ for all $x, y \in R$, then there exists an ideal U of R such that $a \in U \subset Z(R)$ holds.*

2 The Main Result

Theorem 2.1. *Let R be a semiprime ring of characteristic not 2 and $G\colon R \longrightarrow R$ be an additive mapping satisfying the relation*

$$G(x^2) = G(x)x + xD(x), \tag{1}$$

for all $x \in R$ and some derivation D of R. Then G is generalized derivation.

Proof. Replacing x by $x+y$ in (1) we get

$$G(xy+yx) = G(x)y + G(y)x + xD(y) + yD(x), \quad x, y \in R. \tag{2}$$

Replacing y by $xy+yx$ in (2) and using (2) we obtain

$$\begin{aligned} G(x^2y+yx^2)+2G(xyx) = {} & G(x)xy + G(x)yx + G(x)yx + G(y)x^2 + xD(y)x \\ & +yD(x)x + xD(xy+yx) + (xy+yx)D(x), \quad x, y \in R. \end{aligned} \tag{3}$$

On the other hand, replacing x by x^2 in (2) and adding $2G(xyx)$ to both sides we get

$$\begin{aligned} G(x^2y+yx^2)+2G(xyx) = {} & G(x)xy + xD(x)y + G(y)x^2 + x^2D(y) + yxD(x) \\ & +yD(x)x + 2G(xyx), \quad x, y \in R. \end{aligned} \tag{4}$$

Comparing (3) and (4) we obtain

$$G(xyx) = G(x)yx + xD(yx), \quad x, y \in R. \tag{5}$$

Putting $x = x+z$ in (5), we get

$$G(xyz+zyx) = G(x)yz + G(z)yx + xD(yz) + zD(yx), \quad x, y, z \in R. \tag{6}$$

Let $F = G(xyzyx + yxzxy)$, we shall compute it in two different ways. Using (5) we have

$$F = G(x)yzyx + G(y)xzxy + xD(yzyx) + yD(xzxy), \quad x, y, z \in R. \tag{7}$$

Using (6) we have

$$F = G(xy)zyx + G(yx)zxy + xyD(zyx) + yxD(zxy), \quad x, y, z \in R. \tag{8}$$

Comparing (7) and (8) we get

$$\theta(x,y)zyx + \theta(y,x)zxy = 0, \quad x, y, z \in R, \tag{9}$$

where $\theta(x,y)$ stands for $G(xy)-G(x)y-xD(y)$. In the concept of the definition of θ, equation (2) can be rewritten in the form $\theta(x,y) = -\theta(y,x)$. Using this notation in equation (9) we get

$$\theta(x,y)z[x,y] = 0, \quad x,y,z \in R. \tag{10}$$

Using Lemma 1.2 we get

$$\theta(x,y)z[u,v] = 0, \quad x,y,z,u,v \in R. \tag{11}$$

Using Lemma 1.1 we obtain

$$\theta(x,y)[u,v] = 0, \quad x,y,u,v \in R. \tag{12}$$

Now fix $x,y \in R$ and write θ instead of $\theta(x,y)$ to simplify further writing. Using Lemma 1.3 we get the existence of an ideal U such that $\theta \in U \subset Z(R)$ holds. In particular, $b\theta$, $\theta b \in Z(R)$ for all $b \in R$. This gives us

$$x.\theta^2 y = \theta^2 y.x = y\theta^2.x = y.\theta^2 x.$$

This gives us $4G(x.\theta^2 y) = 4G(y.\theta^2 x)$. Now we will compute each side of this equality by using (2) and the above notation.

$$4G(x.\theta^2 y) = 2G(x\theta^2 y + \theta^2 yx) =$$

$$= 2G(x)\theta^2 y + 2xD(\theta^2 y) + 2G(\theta^2 y)x + 2\theta^2 yD(x) =$$

$$= 2G(x)\theta^2 y + G(\theta^2 y + y\theta^2)x + 2xD(\theta^2 y) + 2\theta^2 yD(x) =$$

$$= 2G(x)\theta^2 y + G(\theta)\theta yx + \theta D(\theta)yx + G(y)\theta^2 x + \theta^2 D(y)x + yD(\theta^2)x + 2xD(\theta^2 y) + 2\theta^2 yD(x).$$

So we get

$$4G(x.\theta^2 y) = 2G(x)\theta^2 y + G(\theta)\theta yx + \theta D(\theta)yx + G(y)\theta^2 x + \theta^2 D(y)x + yD(\theta^2)x + 2xD(\theta^2 y) + 2\theta^2 yD(x), \quad x,y \in R. \tag{13}$$

Moreover,

$$4G(y.\theta^2 x) = 2G(y\theta^2 x + \theta^2 xy) =$$

$$= 2G(y)\theta^2 x + 2yD(\theta^2 x) + 2G(\theta^2 x)y + 2\theta^2 xD(y) =$$

$$= 2G(y)\theta^2 x + G(\theta^2 x + x\theta^2)y + 2yD(\theta^2 x) + 2\theta^2 xD(y) =$$

$$= 2G(y)\theta^2 x + G(\theta)\theta xy + \theta D(\theta)xy + G(x)\theta^2 y + \theta^2 D(x)y + xD(\theta^2)y + 2yD(\theta^2 x) + 2\theta^2 xD(y).$$

So we get

$$4G(y.\theta^2x) = 2G(y)\theta^2x + G(\theta)\theta xy + \theta D(\theta)xy + G(x)\theta^2y + \theta^2D(x)y \\ +xD(\theta^2)y + 2yD(\theta^2x) + 2\theta^2xD(y), \quad x, y \in R. \tag{14}$$

Comparing (13) and (14) and using the following notations

$$\theta yx = \theta y.x = x.\theta y = x\theta y = \theta xy,$$

$$\theta D(\theta)yx = D(\theta)\theta yx = D(\theta)\theta xy = \theta D(\theta)xy,$$

$$x\theta D(\theta)y = D(\theta)x\theta y = D(\theta)\theta xy = D(\theta)\theta yx = \theta yD(\theta)x = y\theta D(\theta)x,$$

we obtain

$$G(x)\theta^2y + x\theta^2D(y) = G(y)\theta^2x + y\theta^2D(x)$$

which gives

$$\phi(x, y)\theta^2 = \phi(y, x)\theta^2, \tag{15}$$

where $\phi(x, y)$ stands for $G(x)y + xD(y)$. On the other hand, we also have $4G(xy\theta^2) = 4G(x\theta.y\theta)$. We will compute each side of this equality by using (2) and the properties of θ, so we get

$$4G(xy\theta^2) = 2G(xy\theta^2 + \theta^2xy) = 2G(xy)\theta^2 + 2xyD(\theta^2) + 2G(\theta^2)xy + 2\theta^2D(xy),$$

which gives

$$4G(xy\theta^2) = 2G(xy)\theta^2 + 2xyD(\theta^2) + 2G(\theta^2)xy + 2\theta^2D(xy), \quad x, y \in R. \tag{16}$$

Moreover,

$$4G(x\theta.y\theta) = 2G(x\theta y\theta + y\theta x\theta) =$$

$$= 2G(\theta x)\theta y + 2\theta xD(\theta y) + 2G(\theta y)\theta x + 2\theta yD(\theta x) =$$

$$= G(x\theta + \theta x)\theta y + 2\theta xD(\theta y) + G(y\theta + \theta y)\theta x + 2\theta yD(\theta x) =$$

$$= G(x)\theta^2y + G(\theta)\theta xy + xD(\theta)\theta y + \theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2x + \\ G(\theta)\theta yx + yD(\theta)\theta x + \theta D(y)\theta x + 2\theta yD(\theta x).$$

So we obtain

$$4G(x\theta.y\theta) = G(x)\theta^2y + G(\theta)\theta xy + xD(\theta)\theta y \\ +\theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2x + G(\theta)\theta yx + yD(\theta)\theta x \\ +\theta D(y)\theta x + 2\theta yD(\theta x), \quad x, y \in R. \tag{17}$$

Comparing (16) and (17), we obtain

$$2G(xy)\theta^2 = \phi(x, y)\theta^2 + \phi(y, x)\theta^2, \quad x, y \in R. \tag{18}$$

Using (15), finally we get $G(xy)\theta^2 = \phi(x, y)\theta^2$. But $\theta(x, y) = G(xy) - \phi(x, y)$ and this means $\theta^3 = 0$ so that

$$\theta^2 R \theta^2 = \theta^4 R = (0),$$

$$\theta R \theta = \theta^2 R = (0),$$

which implies $\theta = 0$, and the proof is complete.

It is clear that if we let the derivation D to be the zero derivation in the above theorem, we get the following result.

Corollary 2.2 ([6] Proposition 1.4). *Let R be a semiprime ring of characteristic not 2 and $T : R \to R$ an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in R$. Then T is a left centralizer.*

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