Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 32, 1581 - 1586

Non-Polynomial Spline Solution of a Singularly-Perturbed Boundary-Value Problems

J. Rashidinia

Department of Mathematics, Islamic Azad University of
Tehran Central Branch, Iran
rashidinia@iust.ac.ir

Z. Mahmoodi

Department of Mathematics, Islamic Azad University
of Abhar Branch, Iran

Abstract

In this article, using non-polynomial cubic spline we develop the classes of methods for the numerical solution of singularly perturbed two-point boundary-value problems. The purposed methods are second-order and fourth-order accurate and applicable to problems both in singular and non-singular cases.Numerical results are given to illustrate the efficiency of our methods and compared with the methods given by different authors.

Mathematics Subject Classification: 65L10

Keywords: Non-polynomial Spline, Singular boundary value problems, Truncation error, second and fourth order methods

1 Introduction

We consider a second order singularly perturbed boundary value problem

$$-\epsilon y'' + P(x)y = f(x), \quad 0 < P(x) < P(1), y(0) = A, y(1) = B, \tag{1}$$

where A, B are given constants and ϵ is a small positive parameter such that $0 < \epsilon \leq 1$ and $P(x), f(x)$ are small bounded real functions.The application of spline for the numerical solution of singularly-perturbed boundary-value problems has been described by many authors [1,5-8]. In the present paper, we describe a three-point formula based on non-polynomial cubic spline. In section 2, we first derive the formulation of our spline function approximation, in section 3, we developed our methods. in section 4, truncation error and classification and finally in section 5 application of methods to two examples are given.

2 Spline function approximations

Let us consider a uniform mesh with knots $\Delta : a = x_0 < x_1 < ... < x_N = b$ where $h = \frac{b-a}{N}$ and $x_k = x_0 + kh,\ k = 0(1)N$. A function $S_\Delta(x,\tau)$ of class $C^2[a,b]$, which interpolates $y(x)$ at the knots $\{x_k\}$, depends on a parameter $\tau > 0$ is called a parametric spline function,and reduces to a cubic spline function in the interval $[x_{k-1}, x_k]$ as $\tau \to 0$. If $S_\Delta(x,\tau)$ is a parametric spline function, then in general in the intervals $[x_{k-1}, x_k]$, we can write

$$S''_\Delta(x) + \tau S_\Delta(x) = (S''_\Delta(x_{k-1}) + \tau S_\Delta(x_{k-1}))(\frac{x_k - x}{h}) + (S''_\Delta(x_k) + \tau S_\Delta(x_k))(\frac{x - x_{k-1}}{h}). \quad (2)$$

Solving the differential equation (2) on $[x_{k-1}, x_k]$ and determining the constants of integration from the interpolatory conditions at x_{k-1} and x_k, we obtain

$$S_\Delta(x,p) = \frac{-h^2}{w^2 \sin(w)}(\frac{(x - x_{k-1})}{h} S''_\Delta(x_k)\sin(w) + \frac{(x_k - x)}{h} S''_\Delta(x_{k-1})\sin(w)$$
$$+\frac{h^2}{w^2}(\frac{(x-x_{k-1})}{h}(S''_\Delta(x_k) + \frac{w^2}{h^2}S_\Delta(x_k)) + \frac{(x_k - x)}{h}(S''_\Delta(x_{k-1}) + \frac{w^2}{h^2}S_\Delta(x_{k-1}))) \quad (3)$$

where $w = h\sqrt{\tau}$, see [2]. By using the continuity of the first derivative at x_k we obtain the following useful spline relation:

$$y_{k+1} - 2y_k + y_{k-1} = h^2(\alpha M_{k+1} + 2\beta M_k + \alpha M_{k-1}), \quad (4)$$

where

$$S_\Delta(x_k,\tau) = y(x_k) = y_k, S''_\Delta(x_k,\tau) = y''(x_k) = M_k,$$

and

$$\alpha = \frac{\frac{w}{\sin(w)} - 1}{w^2} \quad , \quad \beta = \frac{1 - \frac{w\cos(w)}{\sin(w)}}{w^2}$$

3 The method

For a numerical solution of the boundary-value problem (1) the interval [0,1] is divided into a set of grid points with steplength $h = \frac{b-a}{N}$, N being a positive integer the spline approximation on [0,1] that consist of the nodal points

$x_0 = a$, $x_N = b$,$x_k = a + kh$, $k = 1(1)N-1$, then

$$-\epsilon y''(x_k) + P(x_k)y(x_k) = f(x_k) \tag{5}$$

or

$$y''(x_k) = \frac{P(x_k)}{\epsilon}y(x_k) - \frac{1}{\epsilon}f(x_k) \tag{6}$$

by using spline's second derivative we have

$$\begin{cases} M_{k-1} = \frac{P_{k-1}}{\epsilon}y_{k-1} - \frac{1}{\epsilon}f_{k-1} \\ M_k = \frac{P_k}{\epsilon}y_k - \frac{1}{\epsilon}f_k \\ M_{k+1} = \frac{P_{k+1}}{\epsilon}y_{k+1} - \frac{1}{\epsilon}f_{k+1} \end{cases} \tag{7}$$

substitute (5)-(7) in (4) we have

$$(\alpha h^2 P_{k-1} - \epsilon)y_{k-1} + 2(\epsilon + h^2\beta P_k)y_k + (\alpha h^2 P_{k+1} - \epsilon)y_{k+1}$$
$$= h^2(\alpha f_{k-1} + 2\beta f_k + \alpha f_{k+1}), k = 1(1)N-1. \tag{8}$$

Finally we have the following system

$$2(\epsilon + h^2\beta P_1)y_1 + (\alpha h^2 P_2 - \epsilon)y_2 = h^2[\alpha f_0 + 2\beta f_1 + \alpha f_2] - (\alpha h^2 P_0 - \epsilon)A, k = 1$$

$$(\alpha h^2 P_{k-1} - \epsilon)y_{k-1} + 2(\epsilon + h^2\beta P_k)y_k + (\alpha h^2 P_{k+1} - \epsilon)y_{k+1} = h^2(\alpha f_{k-1} + 2\beta f_k + \alpha f_{k+1}),$$
$$2 \leq k \leq N-2, \tag{9}$$

$$(\alpha h^2 P_{N-2} - \epsilon)y_{N-2} + 2(\epsilon + h^2\beta P_{N-1})y_{N-1} = h^2[\alpha f_{N-2} + 2\beta f_{N-1} + \alpha f_N]$$
$$-(\alpha h^2 P_N - \epsilon)B \qquad k = N-1, \tag{10}$$

4 Truncation error

By expanding equation (8) in Taylor's series about x_k we obtain

$$T_k = [-1 + 2(\alpha + \beta)]\epsilon h^2 y''(x_k) + (\alpha - \frac{1}{12})\epsilon h^4 y^{(4)}(x_k)$$
$$+[\frac{\alpha}{12} - \frac{1}{360}]\epsilon h^6 y^{(6)}(x_k) + O(h^8),$$

By choosing different values of α and β provided that $\alpha + \beta = \frac{1}{2}$, we can obtain the classes of second order methods.

Remark 1: if $\alpha = \frac{1}{6}$, $\beta = \frac{1}{3}$, $T_k = O(h^4)$ then the resulting method is reduced to cubic spline that is the second order method.

Remark 2: if $\alpha = \frac{1}{12}, \beta = \frac{5}{12}$, $T_k = O(h^6)$ then the resulting method is the fourth-order method.

5 Numerical illustrations

We have solved the following singularly perturbed boundary value problems.

These examples have been solved using the method (8) for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and different values of h, numerical solutions are computed and compared with the exact solutions at grade points. The maximum absolute errors ($E = \max|\bar{y}_i - y_i|$) in numerical solutions are tabulated in tables 1 and 2.

5.1 Example 1.

$$-\epsilon y'' + y = -cos^2(\pi x) - 2\epsilon\pi^2 cos(2\pi x) y(0) = 0 \quad , \quad y(1) = 0, \tag{11}$$

with the exact solution, $y(x) = \frac{e^{(\frac{x-1}{\sqrt{\epsilon}})} + e^{(\frac{-x}{\sqrt{\epsilon}})}}{1+e^{(\frac{-1}{\sqrt{\epsilon}})}} - cos^2(\pi x)$

This problem has been solved using the method (8) with different values of $N = 16, 32, 64.128.256$ and $\epsilon = \frac{1}{16}, ..., \frac{1}{128}$ the maximum absolute errors in solutions are tabulated in table (1) and compared with the results in [1,6-8] which show the accuracy of our methods.

5.2 Example 2.

$$-\epsilon y'' + (1+x)y = -40(x(x^2-1) - 2\epsilon) y(0) = 0 \quad , \quad y(1) = 0, \tag{12}$$

with the exact solution, $y(x) = 40x(1-x)$

This problem has been solved using the method (8) with different values of $N = 16, 32$ and $\epsilon = 0.1 \times 10^{(-3)}, ..., 0.1 \times 10^{(-9)}$ the maximum absolute errors in solutions are tabulated in table (2) and our results compared with the results in [2-4] the results satisfy the superiority of our methods.

Table 1

The maximum absolute errors $\|E\|$ in solutions of problem 1

our method

ϵ	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
1/16	4.07(−5)	2.53(−6)	1.58(−7)	9.87(−9)	6.17(−10)
1/32	2.00(−5)	1.24(−6)	7.74(−8)	4.83(−9)	3.02(−10)
1/64	5.45(−5)	3.42(−6)	2.14(−7)	1.34(−8)	8.39(−10)
1/128	1.83(−4)	1.22(−5)	7.68(−7)	4.81(−8)	3.01(−9)

Surla and Stojanovic's method [6]

1/16	8.06(−3)	2.02(−3)	5.08(−4)	1.27(−4)	3.17(−5)
1/32	7.11(−3)	1.79(−3)	4.48(−4)	1.12(−4)	2.80(−5)
1/64	6.58(−3)	1.66(−3)	4.15(−4)	1.04(−4)	2.60(−5)
1/128	6.36(−3)	1.61(−3)	4.03(−4)	1.01(−4)	2.52(−5)

Surla and Herceg and Cvekovic's method[7]

1/16	4.14(−3)	1.02(−3)	2.54(−4)	6.35(−5)	1.58(−5)
1/32	3.68(−3)	9.03(−4)	5.61(−5)	1.40(−5)	3.50(−6)
1/64	3.45(−3)	8.40(−4)	2.08(−4)	5.20(−5)	1.30(−5)
1/128	3.43(−3)	8.21(−4)	2.03(−4)	5.06(−5)	1.26(−5)

Surla and Vukoslavcevic's method [8]

1/16	1.20(−4)	7.47(−6)	4.67(−7)	2.90(−8)	4.39(−9)
1/32	1.28(−4)	8.00(−6)	5.00(−7)	3.14(−8)	1.99(−9)
1/64	1.60(−4)	1.00(−5)	6.26(−7)	3.92(−8)	2.31(−9)
1/128	2.344(−4)	1.47(−5)	9.23(−7)	5.77(−8)	3.72(−9)

Kadalbajoo and Bawa's method [1]

1/16	7.09(−3)	1.77(−3)	4.45(−4)	1.11(−4)	2.78(−5)
1/32	5.68(−3)	1.42(−3)	3.55(−4)	8.89(−5)	2.22(−5)
1/64	4.07(−3)	1.01(−3)	2.54(−4)	6.35(−5)	1.58(−5)
1/128	6.97(−3)	1.75(−3)	4.33(−4)	1.08(−4)	2.71(−5)

Table 2

The maximum absolute errors $\|E\|$ in solutions of problem 2

$N = 16$

ϵ	$methodin[2]$	$methodin[3]$	$methodin[4]$	$ourmethod$
0.1(−3)	0.25(−1)	0.26(−1)	0.65(−4)	1.776(−15)
0.1(−4)	0.21(−1)	0.24(−1)	0.36(−4)	1.776(−15)
0.1(−5)	0.70(−2)	0.17(−1)	0.33(−4)	1.776(−15)
0.1(−6)	0.75(−3)	0.69(−2)	0.26(−4)	1.776(−15)
0.1(−7)	0.74(−4)	0.23(−2)	0.20(−4)	1.776(−15)
0.1(−8)	0.67(−5)	0.76(−3)	0.20(−4)	1.776(−15)
0.1(−9)	0.00(−0)	0.24(−3)	0.11(−4)	3.552(−15)

$N = 32$

0.1(−3)	0.64(−2)	0.65(−2)	0.59(−4)	1.776(−15)
0.1(−4)	0.61(−2)	0.64(−2)	0.21(−4)	1.776(−15)
0.1(−5)	0.41(−2)	0.56(−2)	0.35(−4)	1.776(−15)
0.1(−6)	0.77(−3)	0.31(−2)	0.39(−4)	1.776(−15)
0.1(−7)	0.76(−4)	0.12(−2)	0.21(−4)	1.776(−15)
0.1(−8)	0.67(−5)	0.38(−3)	0.21(−4)	1.776(−15)
0.1(−9)	0.00(−0)	0.13(−3)	0.14(−4)	3.552(−15)

ACKNOWLEDGEMENTS. This research is supported by Islamic Azad University of Tehran Central Branch,Iran.

References

[1] M.K. Kalbajoo and R.K. Bawa, Variable-mesh difference scheme for singularly-perturbed boundary-value problem using splines ,*Journal of optimization theory and applications*,(1996), 405-416.

[2] J.J.H. Miller, *On The Convergence ,Uniformly in ϵ ,of Difference Schemes for A Two-point Boundary Singular Perturbation Problem ,In:Numerical Analysis of Singular Perturbation Problems* ,Academic Press ,New York , 1979.

[3] K. Niijima, On a three-point difference scheme for a singular perturbation problem without a first derivative term I ,*Mem.Numer.Math* , **7** (1980), 1-10.

[4] K. Niijima, On a three-point difference scheme for a singular perturbation problem without a first derivative term II ,*Mem.Numer.Math* ,**7** (1980), 11-27.

[5] J. Rashidinia, *Applications of spline to numerical solution of differential equates* ,M.phil dissertation A.M.U.India(1990).

[6] K. Surla, M. Stojanovic and Solving singularly-perturbed boundary-value problems by splines in tension ,*J.comutional and applied mathematics*, **24**, (1988), 355-363.

[7] K. Surla, D. Herceg and L. Cvetkovic, A finally of exponential spline difference scheme,Review of research ,faculty of science ,*mathematics series* ,university of Novi Sad, **19**,(1991),12-23.

[8] K. Surla and V.Vukoslavcevic,A spline difference scheme for boundary-value problems with a small parameter,Review of research,Faculty of science,*Mathematics series*,University of Novi Sad ,**25**, (1995),159-166.

Received: November 13, 2006