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# A Note on Umbilical Lightlike Hypersurfaces of Indefinite Sasakian Manifolds

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**Abstract**

In this note, we consider totally contact umbilical lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field. Theorem on a geometrical configuration of these hypersurfaces is obtained (Theorem 2.4). A characterization of totally contact geodesic lightlike hypersurfaces with parallel vector field is given (Theorem 2.8).

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**Keywords:** Lightlike Hypersurfaces; Indefinite Sasakian; Indefinite Sasakian Space Form

# 1 Introduction

A $(2n+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *indefinite Sasakian manifold* if it admits an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$ of rank $2n$, $\xi$ is a vector field, and $\eta$ is a 1-form, satisfying

$$\begin{aligned}
&\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \overline{\phi} = 0,\ \overline{\phi}\xi = 0,\ \eta(\overline{X}) = \overline{g}(\xi, \overline{X}),\\
&\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}), (\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}),\\
&(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X},\ \overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X}),\ \forall\, \overline{X}, \overline{Y} \in \Gamma(T\overline{M}) \quad (1)
\end{aligned}$$

where $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$.

A plane section $\sigma$ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to $\xi$. The sectional curvature of a $\overline{\phi}$-section $\sigma$ is called a $\overline{\phi}$-sectional curvature. A Sasakian manifold $\overline{M}$ with constant $\overline{\phi}$-sectional curvature $c$, $\overline{M}$ is said to be a *Sasakian space form* and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by [5]

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c+3}{4}\left(\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\right) + \frac{c-1}{4}\Big(\eta(\overline{X})\eta(\overline{Z})\overline{Y}$$
$$-\eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X}$$
$$-\overline{g}(\overline{\phi}\,\overline{X},\overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\Big)\,,\ \overline{X},\ \overline{Y},\ \overline{Z} \in \Gamma(T\overline{M}). \tag{2}$$

Let $(\overline{M},\overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index $s$, $0 < s < 2n+1$ and let $(M,g)$ be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. $M$ is a lightlike hypersurface of $\overline{M}$ if $g$ is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on $M$ [4]. A complementary bundle of $TM^{\perp}$ in $TM$ is a rank $2n-1$ non-degenerate distribution over $M$. It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M,g,S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

**Theorem 1.1** [4] *Let $(M,g,S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over $M$ such that for any non-zero section $E$ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section $N$ of $N(TM)$ on $\mathcal{U}$ satisfying*

$$g(N,E) = 1 \ \ \textit{and} \quad \overline{g}(N,N) = \overline{g}(N,W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}).$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(\Xi)$ the smooth sections of the vector bundle $\Xi$. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 1.1 we may write down the following decomposition

$$TM = S(TM) \perp TM^{\perp},$$
$$T\overline{M} = TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \tag{3}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M},\overline{g})$, then by using the second decomposition of (3), we have Gauss and Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X,\, Y \in \Gamma(TM), \tag{4}$$
$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall X \in \Gamma(TM),\, V \in \Gamma(N(TM)), \tag{5}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. $\nabla$ is a symmetric linear connection on $M$ called an induced linear connection, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$. $h$ is a $\Gamma(N(TM))$-valued symmetric bilinear form and $A_V$ is the shape operator of $M$ concerning $V$.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 1.1. Then (4) and (5) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)\,N \;\text{ and }\; \overline{\nabla}_X N = -A_N X + \tau(X)N. \tag{6}$$

It is important to mention that the second fundamental form $B$ is independent of the choice of screen distribution, in fact, from (6), we obtain

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E) \;\text{ and }\; \tau(X) = \overline{g}(\overline{\nabla}^{\perp}_X N, E) \quad \forall X,\, Y \in \Gamma(TM|_{\mathcal{U}}).$$

Let $P$ be the projection morphism of $TM$ on $S(TM)$ with respect to the orthogonal decomposition of $TM$. We have

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E \;\text{ and }\; \nabla_X E = -A^*_E X - \tau(X)E, \tag{7}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. $C$, $A^*_E$ and $\nabla^*$ are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection $\nabla$ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \;\; \forall\, X,\; Y \in \Gamma(TM|_{\mathcal{U}}), \tag{8}$$

where $\theta$ is a differential 1-form locally defined on $M$ by $\theta(\cdot) := \overline{g}(N, \cdot)$. Also, we have the following identities,

$$g(A^*_E X, PY) = B(X, PY),\; g(A^*_E X, N) = 0,\; B(X,E) = 0. \tag{9}$$

Finally, using (6), $\overline{R}$ and $R$ are the curvature tensor fields of $\overline{M}$ and $M$ are related as

$$\begin{aligned} \overline{R}(X,Y)Z \;&=\; R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + ((\nabla_X B)(Y,Z) \\ &\quad -(\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z))\,N, \qquad (10) \\ \text{where} \quad &(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \qquad (11) \end{aligned}$$

## 2 Main Results

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Sasakian manifold and $(M, g)$ be its lightlike hypersurface, *tangent to the structure vector field* $\xi$ ($\xi \in TM$)[1]. If $E$ is a local section of $TM^{\perp}$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to $M$. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on $M$ of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$.

[1]Many geometers use to consider $\xi$ tangent to the manifold because in the theory of $CR$ submanifolds the condition $M$ normal to $\xi$ leads to $M$ anti-invariant submanifold (see [6]; Proposition 1.1, p. 43) and the condition $\xi$ oblique gives very complicated embedding equations

This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as vector subbundle. We consider local section $N$ of $N(TM)$. Since $\overline{g}(\overline{\phi}\,N, E) = -\overline{g}(N, \overline{\phi}\,E) = 0$, we deduce that $\overline{\phi}\,N$ is also tangent to $M$ and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}\,N, N) = 0$, we see that the components of $\overline{\phi}\,N$ with respect to $E$ vanishes. Thus $\overline{\phi}\,N \in \Gamma(S(TM))$. From (1), we have $\overline{g}(\overline{\phi}\,N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

It is known [3] that if $M$ is tangent to the structure vector field $\xi$, then, $\xi$ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution $D_0$ of rank $2n-4$ on $M$ such that [5]

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp \langle\xi\rangle, \tag{12}$$

where $\langle\xi\rangle$ is the distribution spanned by $\xi$, thats is, $\langle\xi\rangle = Span\{\xi\}$. It is easy to check that the distribution $D_0$ is invariant under $\overline{\phi}(D_0) = D_0$.

Moreover, from (3) and (12) we obtain the decomposition

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp TM^{\perp}, \tag{13}$$

$$T\overline{M} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp (TM^{\perp} \oplus N(TM)) \tag{14}$$

Now, we consider the distributions on $M$,

$$D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0, \quad D' := \overline{\phi}(N(TM)). \tag{15}$$

Then $D$ is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{16}$$

Let us consider the local lightlike vector fields $U := -\,\overline{\phi}\,N$, $V := -\,\overline{\phi}\,E$. Then, from (16), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)\,U$, where $R$ and $Q$ are the projection morphisms of $TM$ into $D$ and $D'$, respectively, and $u$ is a differential 1-form locally defined on $M$ by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to $X$ and (1) (note that $\overline{\phi}^2\,N = -N$), we obtain $\overline{\phi}\,X = \phi\,X + u(X)\,N$, where $\phi$ is a tensor field of type $(1,1)$ defined on $M$ by $\phi\,X := \overline{\phi}\,RX$ and we also have $\phi^2\,X = -\,X + \eta(X)\xi + u(X)\,U$, $\forall\,X \in \Gamma(TM)$. Now applying $\phi$ to $\phi^2\,X$ and since $\phi U = 0$, we obtain $\phi^3 + \phi = 0$, which shows that $\phi$ is an $f$-structure [4] of constant rank. By using (1) we derive $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where $v$ is 1-form locally defined on $M$ by $v(\cdot) = g(U, \cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \tag{17}$$

From direct calculations, we have the following useful identities

$$\nabla_X\xi = -\phi X, \tag{18}$$

$$
\begin{aligned}
B(X,\xi) &= -u(X), && (19)\\
B(X,V) &= u(A^*_E X), && (20)\\
B(X,U) &= C(X,V) = u(A_N X), && (21)\\
(\nabla_X u)Y &= -B(X,\phi Y) - u(Y)\tau(X), && (22)\\
(\nabla_X \phi)Y &= g(X,Y)\xi - \eta(Y)X - B(X,Y)U + u(Y)A_N X. && (23)
\end{aligned}
$$

**Proposition 2.1** *Let $M$ be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. The Lie derivative with respect to the vector field $V$ is given by, for any $X$, $Y \in \Gamma(TM)$*

$$(L_V g)(X,Y) = X.u(Y) + Y.u(X) + u([X,Y]) - 2u(\nabla_X Y). \tag{24}$$

**Proof:** From a straightforward calculation, we complete the proof. □

A submanifold $M$ is said to be totally contact umbilical lightlike hypersurface of the a semi-Riemannian manifold $\overline{M}$ if the second fundemental form $h$ of $M$ satisfies

$$h(X,Y) = (g(X,Y) - \eta(X)\eta(Y))\, H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi) \tag{25}$$

for any $X$, $Y \in \Gamma(TM)$, where $H$ is a normal vector field on $M$ (that is $H = \lambda N$, $\lambda$ is a smooth function on $\mathcal{U} \subset M$). The notion of totally contact umbilical submanifolds of Sasakian manifolds corresponds to those of totally umbilical submanifolds of Kählerian manifolds (see [6] for more details). The totally contact umbilical condition (25) can be rewritten as,

$$h(X,Y) = B(X,Y)N = (B_1(X,Y) + B_2(X,Y))\, N, \tag{26}$$

where $B_1(X,Y) = \lambda\,(g(X,Y) - \eta(X)\eta(Y))$ and $B_2(X,Y) = -\eta(X)u(Y) - \eta(Y)u(X)$, since $h(X,\xi) = -u(X)N$. The covariant derivative of the local second fundamental form $B$ of $M$ is given by

$$(\nabla_X B)(Y,Z) = (\nabla_X B_1)(Y,Z) + (\nabla_X B_2)(Y,Z),\ \ \forall X,\, Y,\, Z \in \Gamma(TM). \tag{27}$$

If the $\lambda = 0$ (that is $B_1 = 0$), then the lightlike hypersurface $M$ is said to be totally contact geodesic. The notion of totally contact geodesic submanifolds of Sasakian manifolds corresponds to that of totally geodesic submanifolds of Kaehlerian manifolds.

In the sequel, we need the following lemmas.

**Lemma 2.2** *Let $(M,g)$ be a lightlike hypersurface of an indefinite Sasakian space form $(\overline{M},\overline{g})$ with $\xi \in TM$. For any $X$, $Y \in \Gamma(TM)$*

$$g(\nabla_X V, Y) + u(A^*_E X)\theta(Y) = -B(X,\phi Y) - \tau(X)u(Y). \tag{28}$$

**Proof:** By straightforward calculation and also by using (6) and (7)

$$\begin{aligned} g(\nabla_X V, Y) &= -g((\nabla_X \phi)E, Y) - g(\phi \nabla_X E, Y) \\ &= -g(\overline{\phi}\nabla_X E, Y) + u(\nabla_X E)\theta(Y) \\ &= -g(A^*_E X, \overline{\phi}Y) - \tau(X) g(E, \overline{\phi}Y) - u(A^*_E X)\theta(Y) \\ &= -B(X, \phi Y) - \tau(X)u(Y) - u(A^*_E X)\theta(Y), \end{aligned}$$

which completes the proof. □

**Lemma 2.3** *Let $(M, g)$ be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, $\forall\, X,\, Y,\, Z \in \Gamma(TM)$*

$$\begin{aligned} &(\nabla_X B_1)(Y, Z) = \lambda(B(X, Y)\theta(Z) + B(X, Z)\theta(Y)) + \lambda\eta(Z)(u(X)\theta(Y) \\ &+g(\phi X, Y)) + \lambda\eta(Y)(u(X)\theta(Z) + g(\phi X, Z)) + (g(Y, Z) - \eta(Y)\eta(Z))(X.\lambda) \quad (29) \\ &(\nabla_X B_2)(Y, Z) = (u(X)\theta(Y) + g(\phi X, Y))u(Z) + (u(X)\theta(Z) + g(\phi X, Z))u(Y) \\ &+(\tau(X)u(Y) + B(X, \phi Y))\eta(Z) + (\tau(X)u(Z) + B(X, \phi Z))\eta(Y). \quad (30) \end{aligned}$$

**Proof:** The proof follows from straightforward computing and by using the identities (8), (19) and (28). □

**Theorem 2.4** *Let $\overline{M}(c)$ be an indefinite Sasakian space form and $M$ be a totally contact umbilical lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$. Then $c = -3$ ($\overline{M}(c)$ is of constant curvature -3) and $\lambda$ satisfies the partial differential equations*

$$E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0, \tag{31}$$

$$\textit{and} \qquad PX \cdot \lambda + \lambda\tau(PX) = 0, \;\; \forall\, X \in \Gamma(TM). \tag{32}$$

**Proof:** Let $M$ be a totally contact geodesic lightlike hypersurface. The direct calculation of the right hand side in (10) shows that, for any $X,\, Y \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_X B_1)(Y, Z) &- (\nabla_Y B_1)(X, Z) + (\nabla_X B_2)(Y, Z) - (\nabla_Y B_2)(X, Z) \\ &= \frac{c-1}{4}\left(\overline{g}(\overline{\phi}Y, Z)u(X) - \overline{g}(\overline{\phi}X, Z)u(Y) - 2\overline{g}(\overline{\phi}X, Y)u(Z)\right) \\ &+ \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \end{aligned} \tag{33}$$

From (29) and (30), (33) becomes

$$\begin{aligned} \lambda \;\; &(B(X, Y)\theta(Z) + B(X, Z)\theta(Y)) + \lambda\eta(Z)\left(u(X)\theta(Y) + g(\phi X, Y)\right) \\ &+ \lambda\eta(Y)\left(u(X)\theta(Z) + g(\phi X, Z)\right) + \left(g(Y, Z) - \eta(Y)\eta(Z)\right) X.\lambda \\ &+ \left(u(X)\theta(Y) + g(\phi X, Y)\right)u(Z) + \left(u(X)\theta(Z) + g(\phi X, Z)\right)u(Y) \\ &+ \left(\tau(X)u(Y) + B(X, \overline{\phi}Y)\right)\eta(Z) + \left(\tau(X)u(Z) + B(X, \overline{\phi}Z)\right)\eta(Y) \end{aligned}$$

$$\begin{aligned}
&- \lambda\left(B(X,Y)\theta(Z)+B(Y,Z)\theta(X)\right)-\lambda\eta(Z)\left(u(Y)\theta(X)+g(\phi Y,X)\right)\\
&-\lambda\eta(X)\left(u(Y)\theta(Z)+g(\phi Y,Z)\right)-\left(g(X,Z)-\eta(X)\eta(Z)\right)Y.\lambda\\
&-\ \left(u(Y)\theta(X)+g(\phi Y,X)\right)u(Z)-\left(u(Y)\theta(Z)+g(\phi Y,Z)\right)u(X)\\
&-\ \left(\tau(Y)u(X)+B(Y,\overline{\phi}X)\right)\eta(Z)-\left(\tau(Y)u(Z)+B(Y,\overline{\phi}Z)\right)\eta(X)\\
&=\ \frac{c-1}{4}\left(\overline{g}(\overline{\phi}X,Z)u(Y)-\overline{g}(\overline{\phi}Y,Z)u(X)-2\overline{g}(\overline{\phi}Y,X)u(Z)\right)\\
&+\tau(Y)B(X,Z)-\tau(X)B(Y,Z).
\end{aligned}\tag{34}$$

Using (17) and the identity $B(X,\phi Y)=\lambda g(X,\phi Y)$, (34) can be rewritten as

$$\begin{aligned}
&\lambda\left(B(X,Z)\theta(Y)-B(Y,Z)\theta(X)\right)+2\lambda\left(u(X)\theta(Y)+g(\phi X,Y)\right)\eta(Z)\\
+\ &\lambda\left(\eta(Y)u(X)-\eta(X)u(Y)\right)\theta(Z)+\lambda\left(\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)\right)\\
+\ &\left(g(Y,Z)-\eta(Y)\eta(Z)\right)X.\lambda-\left(g(X,Z)-\eta(X)\eta(Z)\right)Y.\lambda\\
+\ &2\left(u(X)\theta(Y)+g(\phi X,Y)\right)u(Z)+\left(u(Y)g(\phi X,Z)-u(X)g(\phi Y,Z)\right)\\
+\ &\left(\tau(X)u(Y)+\lambda g(X,\phi Y)-\tau(Y)u(X)-\lambda g(Y,\phi X)\right)\eta(Z)\\
+\ &\lambda\left(\eta(Y)g(X,\phi Z)-\eta(X)g(Y,\phi Z)\right)+\left(\tau(X)\eta(Y)-\tau(Y)\eta(X)\right)u(Z)\\
=\ &\frac{c-1}{4}\left(\overline{g}(\overline{\phi}Y,Z)u(X)-\overline{g}(\overline{\phi}X,Z)u(Y)-2\overline{g}(\overline{\phi}X,Y)u(Z)\right)\\
&+\tau(Y)B(X,Z)-\tau(X)B(Y,Z).
\end{aligned}\tag{35}$$

Putting $X=E$ in (35), we find

$$\begin{aligned}
-\ &\lambda B(Y,Z)-2\lambda u(Y)\eta(Z)-\lambda\eta(Y)u(Z)+\left(g(Y,Z)-\eta(Y)\eta(Z)\right)(E.\lambda)\\
-\ &3u(Y)u(Z)+\tau(E)\left(u(Y)\eta(Z)+\eta(Y)u(Z)\right)\\
=\ &\frac{3}{4}(c-1)u(Y)u(Z)-\tau(E)B(Y,Z).
\end{aligned}\tag{36}$$

Take $Y=Z=U$ in (36) we obtain $-3u(U)u(U)=\frac{3}{4}(c-1)u(U)u(U)$, that is, $c=-3$. On the other hand, by taking $Y=V$ and $Z=U$ in (36), we have $(B(V,U)=\lambda)$

$$E.\lambda+\lambda\tau(E)-\lambda^2=0.\tag{37}$$

Finally, substituting $X=PX$, $Y=PY$ and $Z=PZ$ into (35) with $c=-3$ and taking into account that $S(TM)$ is nondegenerate, we obtain

$$(PX\cdot\lambda+\lambda\tau(PX))(PY-\eta(PY)\xi)=(PY\cdot\lambda+\lambda\tau(PY))(PX-\eta(PX)\xi).\tag{38}$$

Putting $PX=\xi$ in (38), we have

$$\left(\xi\cdot\lambda+\lambda\tau(\xi)\right)\left(PY-\eta(PY)\xi\right)=0\tag{39}$$

and by taking $Y=V$, we obtain

$$\xi.\lambda+\lambda\tau(\xi)=0.\tag{40}$$

Writing $PX \in \Gamma(S(TM))$ as $PX = PX' + \eta(PX)\xi$ ($PX' = \sum_i \alpha_i F_i + u(PX)U + v(PX)V$, $\{F_i\}_{1\leq i\leq 2n-4}$ an orthogonal basis of $D_0$) and using (40), we have

$$\begin{aligned} PX\cdot\lambda + \lambda\tau(PX) &= (PX' + \eta(PX)\xi)\cdot\lambda + \lambda\tau(PX' + \eta(PX)\xi) \\ &= PX'\cdot\lambda + \lambda\tau(PX') + \eta(PX)\left(\xi\cdot\lambda + \lambda\tau(\xi)\right) \\ &= PX'\cdot\lambda + \lambda\tau(PX') \end{aligned} \quad (41)$$

which leads to get from (38)

$$(PX'\cdot\lambda + \lambda\tau(PX'))\,PY' = (PY'\cdot\lambda + \lambda\tau(PY'))\,PX'. \quad (42)$$

Now suppose that there exists a vector field $X_0$ on some neighbourhood of $M$ such that $PX'_0\cdot\lambda + \lambda\tau(PX'_0) \neq 0$ at some point $p$ in the neighbourhood. Then, from (42) it follows that all vectors of the fibre $(S(TM) - \langle\xi\rangle)_p \subset S(TM)_p$ are collinear with $(PX'_0)_p$. This contradicts $\dim(S(TM) - \langle\xi\rangle)_p > 1$. This implies (32). $\square$

The equations are similar to those of the indefinite Kählerian case (see [4] for details). However, there are non trivial differences arising in the details of the proof of our Theorem.

From Theorem 2.4 we obtain

**Corollary 2.5 (to Theorem 2.4)** *There exist no totally contact umbilical lightlike real hypersurfaces of indefinite Sasakian space forms $\overline{M}(c)$ ($c \neq -3$) with $\xi \in TM$.*

Also, from (31) and (32), we have

$$\nabla^{\perp}_{E} H = \overline{g}(H, E)^2 N \ \text{ and } \ \nabla^{\perp}_{PX} H = 0,\ \forall X \in \Gamma(TM). \quad (43)$$

**Lemma 2.6** *Let $(M, g)$ be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c = -3)$ with $\xi \in TM$. Then, the mean curvature vector $H$ of $M$ is $S(TM)$-parallel, that is,*

$$\nabla^{\perp}_{PX} H = 0,\ \forall X \in \Gamma(TM). \quad (44)$$

Note that, if we choose, at each point $p \in M$, a connected open set $G$ on $M$ such that $T_pG = S(T_pM)$, then $\nabla^{\perp}_{PX} H = 0$ leads to $H$ is a constant vector field in the direction of the screen distribution $S(TM)$.

A submanifold $M$ is said to be an $\eta$-totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the second fundemental form $h$ of $M$ satisfies

$$h(X, Y) = \lambda\left(g(X, Y) - \eta(X)\eta(Y)\right) N,\ \forall X, Y \in \Gamma(TM). \quad (45)$$

From this definition, we can deduce that the totally contact umbilical lightlike hypersurface $M$ of $\overline{M}$ is also $\eta$-totally umbilical in the direction of $D \perp \langle\xi\rangle$, since the 1-form $u$ vanishes in that direction.

If $M$ is an $\eta$-totally umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$, we have

$$g((\nabla_X h)(Y,Z),E) = (\nabla_X B_1)(Y,Z) + \lambda\tau(X)\left(g(Y,Z) - \eta(Y)\eta(Z)\right). \quad (46)$$

Putting $Z = \xi$ in (46) and using (29), we obtain

$$\begin{aligned} g((\nabla_X h)(Y,\xi),E) &= (\nabla_X B_1)(Y,\xi) + \lambda\tau(X)\left(g(Y,\xi) - \eta(Y)\eta(\xi)\right) \\ &= \lambda\overline{g}(\overline{\phi}X,Y). \end{aligned} \quad (47)$$

If the second fundamental form $h$ of the lightlike hypersurface $M$ is parallel, then, we have $0 = g((\nabla_X h)(Y,\xi),E) = \lambda\overline{g}(\overline{\phi}X,Y)$ which leads, by taking $X = E$ and $Y = U$, to $\lambda\overline{g}(\overline{\phi}E,U) = 0$, that is $\lambda = 0$. Hence, for any $X$, $Y \in \Gamma(TM)$, $B(X,Y) = 0$. Therefore we have

**Proposition 2.7** *Let $(M,g)$ be an $\eta$-totally umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$ with $\xi \in TM$. If the second fundamental form $h$ of $M$ is parallel, then $M$ is totally geodesic.*

**Theorem 2.8** *Let $(M,g)$ be a totally contact geodesic lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$, with $\xi \in TM$. If the local second fundamental form $B$ of $M$ is parallel, then,*

(i) *The 1-form $\tau$ vanishes identically on $M$.*

(ii) *$\overline{\phi}(TM^\perp)$ is a Killing distribution.*

(iii) *$\xi$ and $E$ are Killing vector fields with respect to the local second fundamental form $B$ of $M$*

For the proof of this Theorem, we need the following two lemmas.

**Lemma 2.9** *Let $M$ be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature $c$, with $\xi \in TM$. If the local second fundamental form $B$ of $M$ is parallel, then,*

$$(L_V g)(X,Y) = \tau(\xi)B(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \quad (48)$$

*Moreover, if $\tau(\xi) \neq 0$, then $M$ is totally geodesic if and only if $\overline{\phi}(TM^\perp)$ is a Killing distribution on $M$.*

**Proof:** Using (29), (18), (19) and (22), after some computations we have, for any $X$, $Y \in \Gamma(TM)$,

$$\begin{aligned} 0 &= (\nabla_\xi B)(X,Y) = L_\xi B(X,Y) + B(\phi X,Y) + B(X,\phi Y) \\ &= -\tau(\xi)B(X,Y) - (L_V g)(X,Y) - u(X)\tau(Y) - u(Y)\tau(X). \end{aligned} \quad (49)$$

On the other hand, using (11), we obtain

$$\begin{aligned} 0 = (\nabla_Y B)(\xi,X) &= -Y.u(X) + B(X,\phi Y) + u(\nabla_Y X) \\ &= -(L_V g)(X,Y) - u(Y)\tau(X), \end{aligned} \quad (50)$$

$$\begin{aligned} \text{and} \quad 0 = (\nabla_X B)(Y,\xi) &= X.B(\xi,Y) - B(\nabla_X Y,\xi) - B(Y,\nabla_X \xi) \\ &= -(L_V g)(X,Y) - u(X)\tau(Y). \end{aligned} \quad (51)$$

So substituting (50) and (51) in (49), we obtain (48). If $\tau(\xi) \neq 0$, the equivalence follows. □

**Lemma 2.10** *Let $M$ be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}(c)$ of constant curvature $c$, with $\xi \in TM$. If the second fundamental form $h$ of $M$ is parallel, then, $\overline{\phi}(TM^\perp)$ is a $D \perp \langle\xi\rangle$-Killing distribution and*

$$(L_E B)(X,Y) = -\tau(E)B(X,Y), \forall\, X,\, Y \in \Gamma(TM). \quad (52)$$

**Proof:** Taking $Z = \xi$ in $(\nabla_Z h)(X,Y) = 0$, we have

$$(L_V g)(X,Y) = -u(X)\tau(Y) - u(Y)\tau(X)$$

and for any $X$, $Y \in \Gamma(D \perp \langle\xi\rangle)$, $(L_V g)(X,Y) = 0$. Also by taking $Z = E$ in $(\nabla_Z h)(X,Y) = 0$, we have

$$(L_E B)(X,Y) = -\tau(E)B(X,Y) - 2B(A^*_E X,Y).$$

On the other hand $0 = \overline{g}((\nabla_X h)(Y,E),E) = B(A^*_E X,Y)$. So, we have $(L_E B)(X,Y) = -\tau(E)B(X,Y)$. □

**Proof of Theorem 2.8:** The local second fundamental form $B$ of $M$ is parallel if $(\nabla_Z B)(X,Y) = 0$, for any $X$, $Y$, $Z \in \Gamma(TM)$. Using (30), we have, for any $X \in \Gamma(TM)$, $0 = (\nabla_X B)(\xi,U) = \tau(X)$. The assertions (ii) and (iii) follow from the lemmas above. □

As the geometry of a lightlike hypersurface depends on the chosen screen distribution, it is important to investigate the relationship between some geometrical objects, studied above, with the change of the screen distributions. In this case, it is well known that the local second fundamental form of $M$ on $\mathcal{U}$ is independent of the choice of the screen distribution [4]. This means that all results of this paper which depend only on $B$ are stable with respect to any change of the screen distribution.

Next, we study the effect of the change of the screen distribution on the Lie derivative (24). The relationship between the second fundamental forms $C$ and $C'$ of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by

$$C'(X, PY) = C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W), \ \forall\, X,\ Y \in \Gamma(TM). \quad (53)$$

where $W = \sum_{i=1}^{2n-1} c_i W_i$ is the characteristic vector field of the screen change and $\omega$ is the dual 1-form of $W$ with respect to the induced metric $g$ of $M$, that is, $\omega(\cdot) = g(W, \cdot)$.

**Proposition 2.11** *The Lie derivatives $L_V$ and $L'_V$ of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: $(L'_V g)(X, Y) = (L_V g)(X, Y) - H(X, Y)$, where $H$ is the bilinear form defined by $H(X, Y) = u(X)B(Y, W) + u(Y)B(X, W)$.*

**Proof:** The proof follows from $\tau'(X) = \tau(X) + B(X, W)$. □

**Theorem 2.12** *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the Lie derivative $L_V$ is unique, that is, $L_V$ is independent of $S(TM)$, if and only if, the second fundamental form $h$ (or equivalently $B$) of $M$ vanishes identically on $M$.*

**Proof:** $(L_V g)(X, Y) = (\nabla_X u)Y + (\nabla_Y u)X$ and using Theorem 2.2 in [4], we complete the proof. □

Note that if the lightlike hypersurface $M$ is totally geodesic, the linear connection $\nabla$ is unique.

**Proposition 2.13** *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. The covariant derivatives $\nabla$ and $\nabla'$ of $h$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any $X,\ Y,\ Z \in \Gamma(TM)$, $\overline{g}((\nabla'_X h)(Y, Z), E) = \overline{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}(X, Y, Z)$, where $\mathcal{L}$ is given by*

$$\mathcal{L}(X, Y, Z) = B(X, Y)B(Z, W) + B(X, Z)B(Y, W) + B(Y, Z)B(X, W).$$

It is easy to check that the parallelism of $h$ is independent of the screen distribution $S(TM)$ $(\nabla' h \equiv \nabla h)$ if and only the second fundamental form $B$ of $M$ vanishes identically on $M$.

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