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Some Cubic Blaschke Products and Quadratic Rational Functions with Siegel Disks

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Abstract

We show that for any given complex number μ with modulus at most one and any given real number α, there exists a cubic Blaschke product such that the point at infinity is its fixed point with multiplier μ and its restriction on the unit circle is a critical circle map with rotation number α. Moreover if the given real number α is irrational of bounded type, then a modified Blaschke product is quasiconformally conjugate to some quadratic rational function with a Siegel disk whose boundary is a quasicircle containing its critical point and the point at infinity is its fixed point with multiplier μ.

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1 Introduction

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with fixed point of multiplier $e^{2\pi i\alpha}$ at the origin, where $\alpha \in [0, 1]$ is irrational. If f is linearizable at the origin, then there exists a local holomorphic change of coordinate $\Phi : \mathbb{D} \to \mathbb{C}$ with $0 = \Phi(0)$ such that $\Phi^{-1} \circ f \circ \Phi(z) = e^{2\pi i\alpha}z$, where $\mathbb{D}$ is the unit disk. The Fatou component Δ of f containing $\Phi(\mathbb{D})$ is called the *Siegel disk* centered at the origin.

For the irrational number α, we consider the continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

of α and then a sequence of rational numbers

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

converges to α, where a_n is a positive integer uniquely determined by α for all $n \in \mathbb{N}$. The irrational number α is a *Diophantine number of order* $\kappa \geq 2$ if there exists $\varepsilon > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^\kappa}$$

for all rational numbers p/q. The class of Diophantine numbers of order κ is denoted by $\mathcal{D}_\kappa$. Diophantine numbers of order 2 are said to be of *bounded type.* The irrational number α is a *Bryuno number* if the sum

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

converges. The class of Bryuno numbers is denoted by $\mathcal{B}$. Note that for $\kappa > 2$, $\mathcal{D}_2 \subsetneq \mathcal{D}_\kappa \subsetneq \mathcal{B}$ and $\mathcal{D}_\kappa$ has full measure on $\mathbb{R}/\mathbb{Z}$ (see [6] or [10]). Bryuno showed that if α is a Bryuno number, then f is linearizable at the origin. Yoccoz showed that for $\lambda = e^{2\pi i\alpha}$ if α is not a Bryuno number, then $P_\lambda(z) = z^2 + \lambda z$ is not linearizable at the origin, that is, P_λ is linearizable at the origin if and only if α is a Bryuno number. Moreover the following theorem holds if α is of bounded type. Refer to [9] or [10].

Theorem 1.1 (Ghys-Douady-Herman-Shishikura-Świątek). *If an irrational number $\alpha \in [0,1]$ is of bounded type and $\lambda = e^{2\pi i\alpha}$, then the boundary of the Siegel disk Δ of P_λ centered at the origin is a quasicircle containing its critical point $-\lambda/2$.*

Moreover if the irrational number α is of bounded type and $\lambda = e^{2\pi i\alpha}$, then the following holds:

(a) (Petersen). The Julia set $J(P_\lambda)$ of P_λ is locally connected and has measure zero.

(b) (McMullen). The Hausdorff dimension of $J(P_\lambda)$ is less than 2.

(c) (Graczyk-Jones). The Hausdorff dimension of $\partial\Delta$ is greater than 1.

Conversely, Petersen showed that if $\partial\Delta$ is a quasicircle containing the finite critical point $-\lambda/2$ of P_λ, then $\alpha \in [0,1]$ is of bounded type. Zakeri extended Theorem 1.1 to the case of cubic polynomials.

Theorem 1.2 (Zakeri, [11]). *Let P be a cubic polynomial with fixed point of multiplier $e^{2\pi i\alpha}$ at the origin. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of P centered at the origin is a quasicircle containing one or both critical points.*

Geyer showed the following theorem which is extended to some polynomials.

Theorem 1.3 (Geyer, [4]). *Let $d \geq 1$ and let $P(z) = e^{2\pi i\alpha} z(1 + z/d)^d$. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of P centered at the origin is a quasicircle containing its critical point $-d/(d+1)$.*

Let $F_{\lambda,\mu}(z) = z(z+\lambda)/(\mu z + 1)$ with $\lambda\mu \neq 1$. The origin and the point at infinity are fixed points of $F_{\lambda,\mu}$ of multiplier λ and μ respectively. In the case that $\mu = 0$, $F_{\lambda,0}(z) = P_\lambda(z)$. Therefore the quadratic rational function $F_{\lambda,\mu}$ is considered as a perturbation of the quadratic polynomial P_λ. In the case that $\lambda = e^{2\pi i\alpha}$ and α is irrational of bounded type, we show the following theorem which is a generalization of Theorem 1.1.

Theorem 1.4. *If an irrational number $\alpha \in [0,1]$ is of bounded type, $\lambda = e^{2\pi i\alpha}$ and $\mu \in \overline{\mathbb{D}}$ with $\lambda\mu \neq 1$, then the boundary of the Siegel disk Δ of $F_{\lambda,\mu}$ centered at the origin is a quasicircle containing its critical point.*

2 Cubic Blaschke products

2.1 Existence of cubic Blaschke products

We consider a cubic Blaschke product

$$B(z) = e^{2\pi i\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right)$$

with $a\bar{b} \neq 1$ and $0 < |a| \leq |b| < \infty$. The derivative B' of B is

$$B'(z) = \frac{e^{2\pi i\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2} \cdot g(z),$$

where

$$g(z) = \bar{a}\bar{b}z^4 - 2(\bar{a}+\bar{b})z^3 + \left\{3 - |ab|^2 + |a+b|^2\right\} z^2 - 2(a+b)z + ab.$$

So multipliers of fixed points $z = 0$ and $z = \infty$ are $\lambda = abe^{2\pi i\theta}$ and $\mu = \bar{a}\bar{b}e^{-2\pi i\theta}$ respectively. Let c_1, c_2, $c_3 = 1/\bar{c}_2$ and $c_4 = 1/\bar{c}_1$ be critical points of B. Since they are solutions of $g(z) = 0$, we obtain that

$$g(z) = \bar{a}\bar{b}(z - c_1)(z - c_2)(z - c_3)(z - c_4)$$

$$= \bar{a}\bar{b}\left\{z^4 - C_3 z^3 + C_2 z^2 - C_1 z + C_0\right\},$$

where

$$C_3 = c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2},$$

$$C_2 = \frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right),$$

$$C_1 = \frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right),$$

$$C_0 = \frac{c_1 c_2}{\bar{c}_1 \bar{c}_2}.$$

Comparing coefficients of two representations of $g(z)$ implies that

$$c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2} = \frac{2(\bar{a} + \bar{b})}{\bar{a}\bar{b}}, \tag{1}$$

$$\frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right) = \frac{3 - |ab|^2 + |a + b|^2}{\bar{a}\bar{b}}, \tag{2}$$

$$\frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right) = \frac{2(a + b)}{\bar{a}\bar{b}}, \tag{3}$$

$$\frac{c_1 c_2}{\bar{c}_1 \bar{c}_2} = \frac{ab}{\bar{a}\bar{b}}. \tag{4}$$

Eliminating c_1 and $\bar{c}_1$ form equations (1), (2) and (4) gives that

$$|a + b|^2 - 2\left(c_2 + \frac{1}{\bar{c}_2}\right)(\bar{a} + \bar{b}) - \left(\frac{\bar{c}_2}{c_2}\right)ab + \left\{\left(c_2 + \frac{1}{\bar{c}_2}\right)^2 - \frac{c_2}{\bar{c}_2}\right\}\bar{a}\bar{b} + 3 - |ab|^2 = 0 \tag{5}$$

and eliminating c_1 and $\bar{c}_1$ form equations (1), (3) and (4) gives that

$$\frac{\bar{c}_2}{c_2}\left(c_2+\frac{1}{\bar{c}_2}\right)ab+2\left(\frac{c_2}{\bar{c}_2}\right)(\bar{a}+\bar{b})=\frac{c_2}{\bar{c}_2}\left(c_2+\frac{1}{\bar{c}_2}\right)\bar{a}\bar{b}+2(a+b). \tag{6}$$

We obtain that

$$|a+b|^2-4e^{2\pi i\varphi}(\bar{a}+\bar{b})-e^{2\pi i(-2\varphi)}ab+3e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+3-|ab|^2=0 \tag{7}$$

and

$$e^{2\pi i(-2\varphi)}ab+e^{2\pi i\varphi}(\bar{a}+\bar{b})=e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+e^{2\pi i(-\varphi)}(a+b) \tag{8}$$

by substituting $c_2=e^{2\pi i\varphi}$ into equations (5) and (6). Eliminating ab form equations (7) and (8) gives that

$$|a+b|^2-3e^{2\pi i\varphi}(\bar{a}+\bar{b})-e^{2\pi i(-\varphi)}(a+b)+2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+3-|ab|^2=0. \tag{9}$$

Let $\zeta=a+b$. Then

$$|\zeta|^2-3e^{2\pi i\varphi}\bar{\zeta}-e^{2\pi i(-\varphi)}\zeta+2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+3-|ab|^2=0. \tag{10}$$

The real part of the left side of the equation (10) is

$$x^2+y^2-4x\cos 2\pi\varphi-4y\sin 2\pi\varphi+2r\cos 2\pi(2\varphi+\theta+\omega)+3-r^2=0 \tag{11}$$

and the imaginary part of the left side of the equation (10) is

$$y\cos 2\pi\varphi-x\sin 2\pi\varphi+r\sin 2\pi(2\varphi+\theta+\omega)=0, \tag{12}$$

where $\zeta=x+iy$ and $\mu=\bar{a}\bar{b}e^{-2\pi i\theta}=re^{2\pi i\omega}$. One of the solutions of simultaneous equations (11) and (12) are

$$x=-r\cos 2\pi(3\varphi+\theta+\omega)+3\cos 2\pi\varphi$$

and

$$y=-r\sin 2\pi(3\varphi+\theta+\omega)+3\sin 2\pi\varphi,$$

and hence

$$\zeta=re^{2\pi i(3\varphi+\theta+\omega+1/2)}+3e^{2\pi i\varphi}$$

satisfies the equation (10). Conversely, we show the following theorem.

Theorem 2.1. *Let $\mu=re^{2\pi i\omega}\in\overline{\mathbb{D}}$ and let $a=a(\theta,\varphi)$ and $b=b(\theta,\varphi)$ with $|a|\le|b|$ be complex numbers satisfying relations $a+b=re^{2\pi i(3\varphi+\theta+\omega+1/2)}+3e^{2\pi i\varphi}$ and $ab=re^{-2\pi i(\theta+\omega)}$, that is, a and b are the solutions of the equation*

$$\xi^2-\left\{re^{2\pi i(3\varphi+\theta+\omega+1/2)}+3e^{2\pi i\varphi}\right\}\xi+re^{-2\pi i(\theta+\omega)}=0, \tag{†}$$

where $(\theta,\varphi)\in[0,1]^2$. Then the following holds:

(a) *In the case that* $r = 0$*, solutions of the equation* (†) *are* $a = 0$ *and* $b = 3e^{2\pi i\varphi}$.

(b) *In the case that* $0 < r < 1$*, the equation* (†) *does not have double roots. Moreover* $0 < |a| < 1 < |b| < \infty$.

(c) *In the case that* $r = 1$ *and* $2\varphi + \theta + \omega \equiv 0 \pmod 1$*, the equation* (†) *has double roots and* $a = b = e^{2\pi i\varphi}$.

(d) *In the case that* $r = 1$ *and* $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$*, the equation* (†) *does not have double roots. Moreover* $0 < |a| < 1 < |b| < \infty$.

(e) *In the case* (a), (b) *or* (d),

$$B(z) = B_{\theta,\varphi}(z) = e^{2\pi i\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right)$$

is a Blaschke product of degree 3 *and the point at infinity is a fixed point of* B *with multiplier* μ*. Moreover* $z = e^{2\pi i\varphi}$ *is a critical point of* B *and the other two critical points of* B *are in* $\widehat{\mathbb{C}} \setminus \mathbb{T}$*, where* $\mathbb{T}$ *is the unit circle. In this case,* $B|_{\mathbb{T}} : T \to \mathbb{T}$ *is a homeomorphism.*

Proof of (a). It is clear.

Proof of (b). Since $0 < r < 1$, we obtain that $|a + b| \geq |r - 3| > 2$ and $|a||b| < 1$. In this case, either $0 < |a| < 1 \leq |b| < \infty$ or $0 < |a| \leq |b| \leq 1$ hold. If $0 < |a| \leq |b| \leq 1$, then

$$2 < |a + b| \leq |a| + |b| \leq 2.$$

This is a contradiction and hence the situation $0 < |a| < 1 \leq |b| < \infty$ happens. If $|b| = 1$, then

$$2 < |a + b| \leq |a| + |b| = |a| + 1 \leq 2.$$

This is a contradiction. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (c). If $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$, then $a + b = 2e^{2\pi i\varphi}$ and $ab = e^{2\pi i\cdot 2\varphi}$. Therefore the equation (†) has double roots and $a = b = e^{2\pi i\varphi}$.

Proof of (d). Since $|a||b| = r = 1$, either $0 < |a| < 1 < |b| < \infty$ or $|a| = |b| = 1$ hold. If $|a| = |b| = 1$, then

$$2 = |r - 3| \leq |a + b| \leq |a| + |b| = 2$$

and hence $|a + b| = 2$. On the other hand,

$$|a + b| = \left|re^{2\pi i(3\varphi+\theta+\omega+1/2)} + 3e^{2\pi i\varphi}\right| = \left|e^{2\pi i(2\varphi+\theta+\omega)} - 3\right|.$$

So we obtain that $|e^{2\pi i(2\varphi+\theta+\omega)} - 3| = 2$ and hence $2\varphi + \theta + \omega \equiv 0 \pmod 1$. This contradicts that $2\varphi+\theta+\omega \not\equiv 0 \pmod 1$. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (e). In the case that $r = 0$, critical points of B are $z = 0$, $e^{2\pi i\varphi}$ and ∞. Therefore the assertion holds. We consider the case that $0 < r \leq 1$ below. Let

$$f(z) = f_{\theta,\varphi}(z) = \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right) = \frac{1}{\bar{a}\bar{b}} \cdot \frac{z^2 - (a+b)z + ab}{z^2 - \left(\dfrac{\bar{a}+\bar{b}}{\bar{a}\bar{b}}\right)z + \dfrac{1}{\bar{a}\bar{b}}}.$$

The necessary and sufficient condition that the degree of the Blaschke product B be 3 is that the function f be not constant. So the necessary and sufficient condition that the degree of the Blaschke product B be 1 is that the function f be constant, that is,

$$\frac{z^2 - (a+b)z + ab}{z^2 - \left(\dfrac{\bar{a}+\bar{b}}{\bar{a}\bar{b}}\right)z + \dfrac{1}{\bar{a}\bar{b}}} = 1 \tag{13}$$

for all $z \in \mathbb{C}$. Comparing coefficients of the numerator and the denominator of (13) implies that $\bar{a}\bar{b}(a+b) = \bar{a}+\bar{b}$ and $|ab| = 1$. In the case that $0 < r < 1$, the degree of the Blaschke product B is 3 since $|ab| = r < 1$. In the case that $r = 1$,

$$\bar{a}\bar{b}(a+b) - \left(\bar{a}+\bar{b}\right) = -e^{-2\pi i(3\varphi+\theta+\omega)}\left(e^{2\pi i(2\varphi+\theta+\omega)} - 1\right)^3.$$

Therefore in the case $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$, the degree of the Blaschke product B is 3. It is clear that the point at infinity is a fixed point of B with multiplier μ. Moreover it is clear that $g(e^{2\pi i\varphi}) = 0$ and hence $z = e^{2\pi i\varphi}$ is a critical point of B, where

$$B'(z) = \frac{e^{2\pi i\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2} \cdot g(z)$$

and

$$g(z) = \bar{a}\bar{b}z^4 - 2(\bar{a}+\bar{b})z^3 + \left\{3 - |ab|^2 + |a+b|^2\right\}z^2 - 2(a+b)z + ab.$$

Finally we show that the other two critical points of B are in $\widehat{\mathbb{C}} \setminus \mathbb{T}$. we factor $r^{-1}e^{-2\pi i(\theta+\omega)}g(z)$ as

$$\frac{1}{r} \cdot e^{-2\pi i(\theta+\omega)} \cdot g(z) = \left(z - e^{2\pi i\varphi}\right)^2 \cdot h(z),$$

where

$$h(z) = z^2 + 2e^{2\pi i\varphi}\left\{e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} - \frac{3}{r}e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right\}z + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)}.$$

Let

$$h_1(z) = 2e^{2\pi i\varphi}\left\{e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} - \frac{3}{r}e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right\}z$$

and

$$h_2(z) = z^2 + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)}.$$

For $z \in \mathbb{T}$, $|h_2(z)| \leq 2$. In the case that $0 < r < 1$, we obtain that

$$|h_1(z)| \geq 2\left|\frac{3}{r} - 1 - 1\right| > 2$$

on $\mathbb{T}$. In the case that $r = 1$, we obtain that

$$\begin{aligned}|h_1(z)| &= 2\left|e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} - 3e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right| \\ &= 2\left|\left\{e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right\}^2 - 5e^{-2\pi i(2\varphi+\theta+\omega)}\right| \\ &\geq 2\left(5 - \left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2\right) > 2\end{aligned}$$

on $\mathbb{T}$, since $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$. By Rouché's theorem, the number of roots of $h(z) = h_1(z) + h_2(z)$ on $\mathbb{D}$ is one since $|h_1(z)| > 2 \geq |h_2(z)|$ on $\mathbb{T}$ and the number of roots of $h_1(z)$ on $\mathbb{D}$ is one. So one of critical points of B other than $z = e^{2\pi i\varphi}$ is in $\mathbb{D}$. Since critical points of a Blaschke product are symmetric with respect to the unit circle, the other one critical point of B is in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. In this case, the inverse image $B^{-1}(\mathbb{T})$ of the unit circle $\mathbb{T}$ is the union of $\mathbb{T}$ and a figure eight which crosses at $z = e^{2\pi i\varphi}$. Refer to Figure 1. Therefore $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. □

Remark 2.2. Two complex numbers $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ satisfy that

$$a(\theta + 1, \varphi) = a(\theta, \varphi) = a(\theta, \varphi + 1)$$

and

$$b(\theta + 1, \varphi) = b(\theta, \varphi) = b(\theta, \varphi + 1).$$

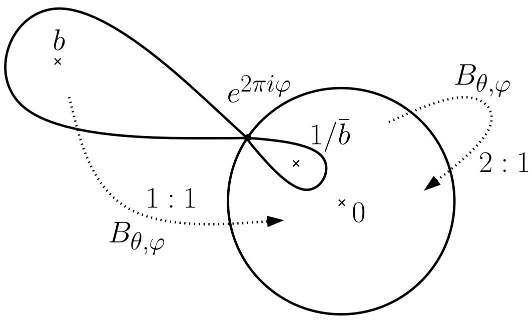

Figure 1: The inverse image $B_{\theta,\varphi}^{-1}(\mathbb{T})$ of the unit circle $\mathbb{T}$.

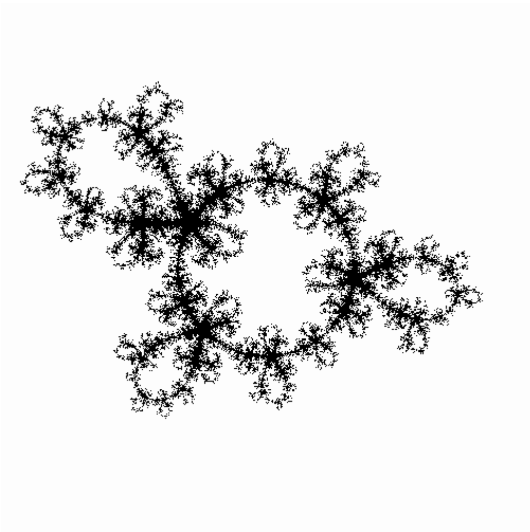

Figure 2: The Julia set of some cubic Blaschke product $B_{\theta,\varphi}$.

2.2 Rotation numbers of cubic Blaschke products

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism and let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f via $x \mapsto e^{2\pi i x}$ which satisfies $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for all $x \in \mathbb{R}$. A lift $\tilde{f}$ of f is unique up to addition of an integer constant. The rotation number $\rho(\tilde{f})$ of $\tilde{f}$ is defined as

$$\rho(\tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(x)}{n},$$

which is independent of $x \in \mathbb{R}$. The rotation number $\rho(f)$ is defined as the residue class of $\rho(\tilde{f})$ modulo $\mathbb{Z}$. Poincaré showed that the rotation number is rational with denominator q if and only if f has a periodic point with period q. The following theorem is important (see [5]).

Theorem 2.3. *Let $\mathcal{F}$ be the set of all orientation preserving homeomorphisms form the unit circle onto itself with the topology of uniform conver-*

gence. Then the rotation number function $\rho : \mathcal{F} \to \mathbb{R}/\mathbb{Z}$ defined as $f \mapsto \rho(f)$ is continuous.

If the cubic Blaschke product $B_{\theta,\varphi}$ as in Theorem 2.1 is an orientation preserving homeomorphism on $\mathbb{T}$, the rotation number function $(\theta, \varphi) \mapsto \rho(B_{\theta,\varphi}|_{\mathbb{T}})$ is continuous. In order to show that $B_{\theta,\varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism, we show the following lemma.

Lemma 2.4. *Let $a(\theta, \varphi)$ and $b(\theta, \varphi)$ be as in Theorem 2.1. Then for any $(\theta, \varphi) \in [0, 1]^2$, a loop $\Gamma_1[\theta, \varphi] : [0, 1] \to \mathbb{T}$ defined as*

$$\Gamma_1[\theta, \varphi](x) = \left(\frac{e^{2\pi i x} - a(\theta, \varphi)}{1 - \overline{a(\theta, \varphi)} e^{2\pi i x}} \right) \left(\frac{e^{2\pi i x} - b(\theta, \varphi)}{1 - \overline{b(\theta, \varphi)} e^{2\pi i x}} \right)$$

is homotopic to a constant loop $x \mapsto e^{2\pi i \cdot 2\varphi}$.

Proof. Note that $\Gamma_1[\theta, \varphi](x) = e^{2\pi i \cdot 2\varphi}$ for all $x \in \mathbb{R}$ if $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$. Let

$$H_1[\theta, \varphi](x, t) = \left(\frac{e^{2\pi i \{(1-t)x + t\varphi\}} - a(\theta, \varphi)}{1 - \overline{a(\theta, \varphi)} e^{2\pi i \{(1-t)x + t\varphi\}}} \right) \left(\frac{e^{2\pi i \{(1-t)x + t\varphi\}} - b(\theta, \varphi)}{1 - \overline{b(\theta, \varphi)} e^{2\pi i \{(1-t)x + t\varphi\}}} \right).$$

Then $H_1[\theta, \varphi](x, 0) = \Gamma_1[\theta, \varphi](x)$ and $H_1[\theta, \varphi](x, 1) = e^{2\pi i \cdot 2\varphi}$. Therefore $H_1[\theta, \varphi]$ is a homotopy between the loop $\Gamma_1[\theta, \varphi]$ and the constant loop $\theta \mapsto e^{2\pi i \cdot 2\varphi}$. $\square$

The following two lemmas play important roles in the proof of Theorem 2.7.

Lemma 2.5. *Let $a(\theta, \varphi)$ and $b(\theta, \varphi)$ be as in Theorem 2.1. Then for any $z \in \mathbb{T}$ and $\varphi \in [0, 1]$, a loop $\Gamma_2[z, \varphi] : [0, 1] \to \mathbb{T}$ defined as*

$$\Gamma_2[z, \varphi](\theta) = \left(\frac{z - a(\theta, \varphi)}{1 - \overline{a(\theta, \varphi)} z} \right) \left(\frac{z - b(\theta, \varphi)}{1 - \overline{b(\theta, \varphi)} z} \right)$$

is homotopic to a constant loop $\theta \mapsto e^{2\pi i \cdot 2\varphi}$.

Proof. Note that $\Gamma_2[e^{2\pi i \varphi}, \varphi](\theta) = e^{2\pi i \cdot 2\varphi}$ for all $\theta \in [0, 1]$ and hence $\Gamma_2[e^{2\pi i \varphi}, \varphi]$ is a constant loop $e^{2\pi i \cdot 2\varphi}$. Let

$$H_2[z, \varphi](\theta, t) = \left(\frac{z - a(\theta, \varphi, t)}{1 - \overline{a(\theta, \varphi, t)} z} \right) \left(\frac{z - b(\theta, \varphi, t)}{1 - \overline{b(\theta, \varphi, t)} z} \right),$$

where

$$a(\theta, \varphi, t) = (1 - t) a(\theta, \varphi) + t e^{2\pi i \varphi}$$

and

$$b(\theta, \varphi, t) = (1 - t) b(\theta, \varphi) + t e^{2\pi i \varphi}.$$

Then $H_2[z, \varphi](\theta, 0) = \Gamma_2[z, \varphi](\theta)$ and $H_2[z, \varphi](\theta, 1) = e^{2\pi i \cdot 2\varphi}$. Therefore $H_2[z, \varphi]$ is a homotopy between the loop $\Gamma_2[z, \varphi]$ and the constant loop $\theta \mapsto e^{2\pi i \cdot 2\varphi}$. $\square$

Lemma 2.6. *Let $a(\theta,\varphi)$ and $b(\theta,\varphi)$ be as in Theorem 2.1. Then for any $z \in \mathbb{T}$ and $\theta \in [0,1]$, a loop $\Gamma_3[z,\theta] : [0,1] \to \mathbb{T}$ defined as*

$$\Gamma_3[z,\theta](\varphi) = \left(\frac{z - a(\theta,\varphi)}{1 - \overline{a(\theta,\varphi)}z}\right)\left(\frac{z - b(\theta,\varphi)}{1 - \overline{b(\theta,\varphi)}z}\right)$$

is homotopic to a loop $\varphi \mapsto e^{2\pi i\cdot 2\varphi}$.

Proof. Note that $\Gamma_3[e^{2\pi i\varphi},\theta](\varphi) = e^{2\pi i\cdot 2\varphi}$. Let

$$H_3[z,\theta](\varphi,t) = \left(\frac{z - a(\theta,\varphi,t)}{1 - \overline{a(\theta,\varphi,t)}z}\right)\left(\frac{z - b(\theta,\varphi,t)}{1 - \overline{b(\theta,\varphi,t)}z}\right),$$

where

$$a(\theta,\varphi,t) = (1-t)a(\theta,\varphi) + te^{2\pi i\varphi}$$

and

$$b(\theta,\varphi,t) = (1-t)b(\theta,\varphi) + te^{2\pi i\varphi}.$$

Then $H_3[z,\theta](\varphi,0) = \Gamma_3[z,\theta](\varphi)$ and $H_3[z,\theta](\varphi,1) = e^{2\pi i\cdot 2\varphi}$. Therefore $H_3[z,\theta]$ is a homotopy between the loop $\Gamma_3[z,\theta]$ and the the loop $\varphi \mapsto e^{2\pi i\cdot 2\varphi}$. □

Let

$$\Gamma(x,\theta,\varphi) = \left(\frac{e^{2\pi ix} - a(\theta,\varphi)}{1 - \overline{a(\theta,\varphi)}e^{2\pi ix}}\right)\left(\frac{e^{2\pi ix} - b(\theta,\varphi)}{1 - \overline{b(\theta,\varphi)}e^{2\pi ix}}\right).$$

Then $\Gamma(x,\theta,\varphi) = \Gamma_1[\theta,\varphi](x) = \Gamma_2[e^{2\pi ix},\varphi](\theta) = \Gamma_3[e^{2\pi ix},\theta](\varphi)$. Lemma 2.4 and Lemma 2.5 imply that

$$\arg\left(\Gamma(x+1,\theta,\varphi)\right) = \arg\left(\Gamma(x,\theta,\varphi)\right) = \arg\left(\Gamma(x,\theta+1,\varphi)\right)$$

and Lemma 2.6 implies that

$$\frac{1}{2\pi}\arg\left(\Gamma(x,\theta,\varphi+1)\right) = \frac{1}{2\pi}\arg\left(\Gamma(x,\theta,\varphi)\right) + 2.$$

Theorem 2.7. *Let $\alpha \in [0,1]$ and let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$, $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ be as in Theorem 2.1. Then for the Blaschke product*

$$B_{\theta,\varphi}(z) = e^{2\pi i\theta}z\left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right),$$

$B_{\theta,\varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Moreover

(a) *In the case that $0 \le r < 1$, there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that $\rho(B_{\theta_0,\varphi_0}|_{\mathbb{T}}) = \alpha$.*

(b) *In the case that* $r = 1$, *if* $\alpha + \omega \not\equiv 0 \pmod 1$, *then there exists* $(\theta_0, \varphi_0) \in [0,1]^2$ *such that* $\rho(B_{\theta_0,\varphi_0}|_{\mathbb{T}}) = \alpha$ *and* $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$.

Proof. In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$,

$$B_{\theta,\varphi}(z) = e^{2\pi i(2\varphi+\theta)} z = e^{2\pi i(-\omega)} z.$$

Therefore $B_{\theta,\varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism and its rotation number satisfies that $\rho(B_{\theta,\varphi}|_{\mathbb{T}}) \equiv -\omega \pmod 1$. In the other cases, we consider a lift

$$\widetilde{B}_{\theta,\varphi}(x) = \theta + x + \frac{1}{2\pi} \arg\left(\Gamma(x,\theta,\varphi)\right)$$

of $B_{\theta,\varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ via $x \mapsto e^{2\pi i x}$. By Lemma 2.4,

$$\widetilde{B}_{\theta,\varphi}(x+1) = \theta + x + 1 + \frac{1}{2\pi} \arg\left(\Gamma(x+1,\theta,\varphi)\right) = \widetilde{B}_{\theta,\varphi}(x) + 1$$

for all $x \in \mathbb{R}$. This implies that $B_{\theta,\varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Consequently the rotation number of $\rho(\widetilde{B}_{\theta,\varphi})$ is well defined. By Lemma 2.5, we obtain that $\widetilde{B}^n_{1,\varphi}(x) = \widetilde{B}^n_{0,\varphi}(x) + n$ and hence

$$\rho(\widetilde{B}_{1,\varphi}) = \rho(\widetilde{B}_{0,\varphi}) + 1. \tag{14}$$

Moreover by Lemma 2.6, we obtain that $\widetilde{B}^n_{\theta,1}(x) = \widetilde{B}^n_{\theta,0}(x) + 2n$ and hence

$$\rho(\widetilde{B}_{\theta,1}) = \rho(\widetilde{B}_{\theta,0}) + 2. \tag{15}$$

These two equation (14) and (15) imply that

$$\rho(\widetilde{B}_{1,1}) = \rho(\widetilde{B}_{0,0}) + 3.$$

Therefore in the case that $0 \le r < 1$, there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0}) \pmod 1$$

since the rotation number function function $(\theta,\varphi) \mapsto \rho(B_{\theta,\varphi}|_{\mathbb{T}})$ is continuous. In the case that $r = 1$, if $2\varphi + \theta + \omega \equiv 0 \pmod 1$, then $\rho(B_{\theta,\varphi}|_{\mathbb{T}}) \equiv -\omega \pmod 1$. Hence if $\alpha + \omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0}) \pmod 1$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$. □

Remark 2.8. By theorem 2.1, the degree of B_{θ_0,φ_0} is 3.

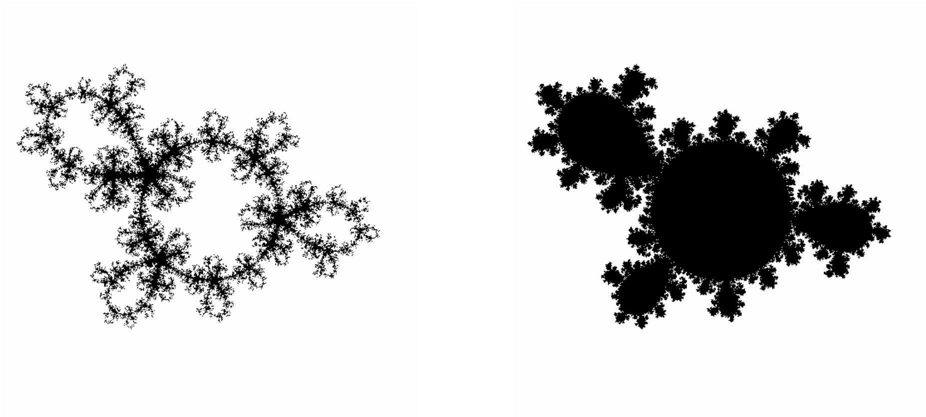

Figure 3: The Julia set of some cubic Blaschke product $B_{\theta,\varphi}$ (left) and "the filled-in Julia set" of some modified Blaschke product $\mathfrak{B}_{\theta,\varphi}$ (right).

3 Quadratic rational functions with Siegel disks

In this section, we show Theorem 1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. The map f is *k-quasisymmetric* if there exists $k \geq 1$ such that

$$\frac{1}{k} \leq \left| \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \right| \leq k$$

for all $x \in \mathbb{R}$ and all $t \geq 0$. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is *k-quasisymmetric* if its lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is k-quasisymmetric. By the theorem of Beurling and Ahlfors, any k-quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is extended to a K-quasiconformal map $F : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$, where $\mathbb{H}$ is the upper half plain (More precisely $F : \mathbb{C} \to \mathbb{C}$). The dilatation K of F depends only on k. Therefore if a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is k-quasisymmetric, then we can extend h to a K-quasiconformal map $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose dilatation depends only on k.

Theorem 3.1 (Herman-Świątek). *The rotation number $\rho(f)$ of a real analytic orientation preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is of constant type if and only if f is quasisymmetrically linearizable, that is, there exits a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i \rho(f)} z$.*

Let $F_{\lambda,\mu}(z) = z(z+\lambda)/(\mu z + 1)$ with $\lambda\mu \neq 1$. Any quadratic rational function with fixed points of multipliers λ and μ with $\lambda\mu \neq 1$ is conjugate to $F_{\lambda,\mu}$ (see [7]).

Proof of Theorem 1.4. By Theorem 2.7, there exist $(\theta, \varphi) \in [0,1]^2$ such that the degree of $B_{\theta,\varphi}$ is 3 and $\rho(B_{\theta,\varphi}|_{\mathbb{T}}) = \alpha$. Since α is of bounded type, there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ B_{\theta,\varphi}|_{\mathbb{T}} \circ h^{-1}(z) = R_\alpha(z) = e^{2\pi i\alpha} z$. By the theorem of Beurling and Ahlfors, h has a

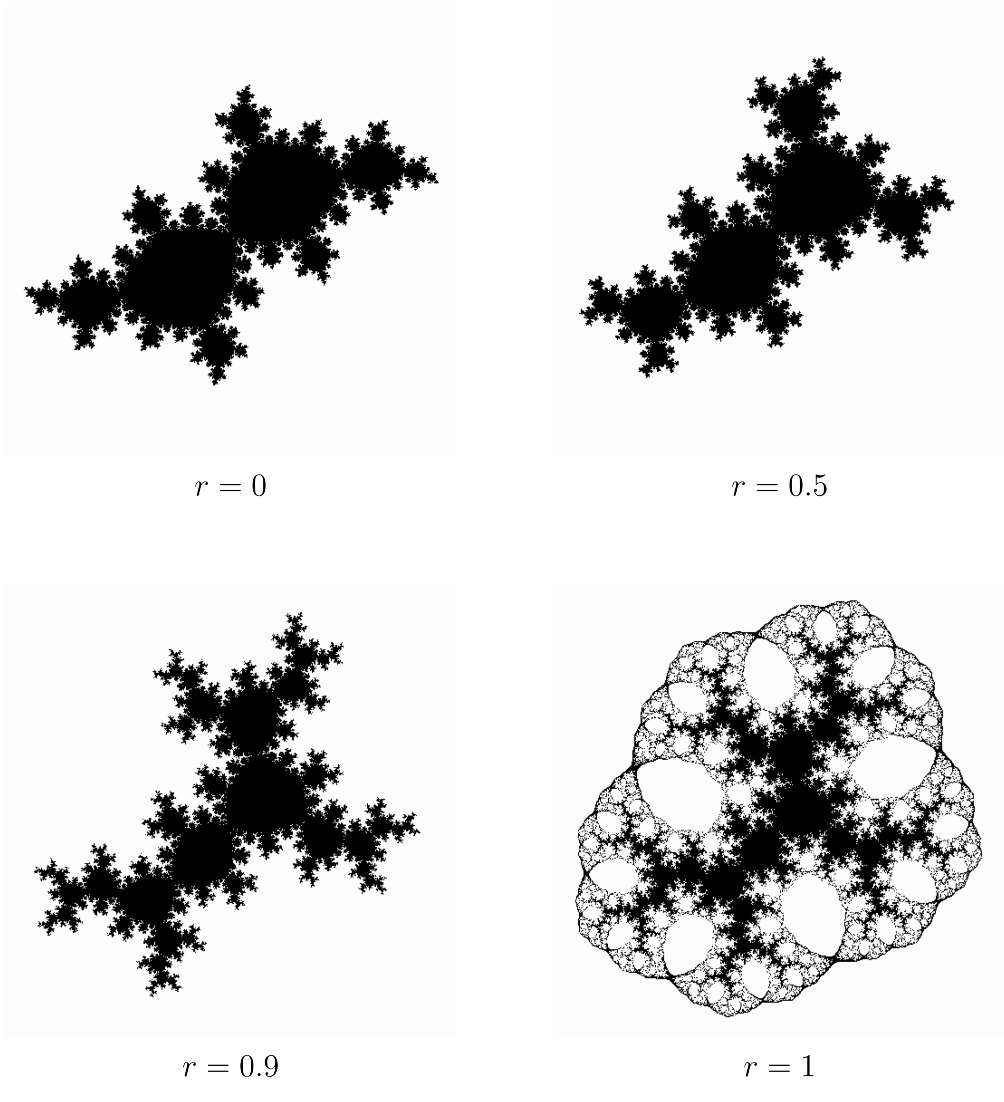

Figure 4: Golden Siegel disks of $F_{\lambda,\mu}$ centered at the origin, where $\lambda = e^{2\pi i\cdot(\sqrt{5}-1)/2}$ and $\mu = re^{2\pi i\cdot(\sqrt{5}-1)/2}$. In the case $r = 1$, the point at infinity is the center of another golden Siegel disk.

quasiconformal extension $H:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ with $H(0)=0$. We define a new map $\mathfrak{B}_{\theta,\varphi}$ as

$$\mathfrak{B}_{\theta,\varphi}=\begin{cases} B_{\theta,\varphi} & \text{on } \widehat{\mathbb{C}}\setminus\mathbb{D},\\ H^{-1}\circ R_\alpha\circ H & \text{on } \mathbb{D}.\end{cases}$$

The map $\mathfrak{B}_{\theta,\varphi}$ is quasiregular on $\widehat{\mathbb{C}}$ since $\mathbb{T}$ is an analytic curve. Moreover $\mathfrak{B}_{\theta,\varphi}$ is a degree 2 branched covering of $\widehat{\mathbb{C}}$. We define a conformal structure $\sigma_{\theta,\varphi}$ as

$$\sigma_{\theta,\varphi}=\begin{cases} H^*\sigma_0 & \text{on } \mathbb{D},\\ \left(\mathfrak{B}^n_{\theta,\varphi}\right)^*\sigma_0 & \text{on } \mathfrak{B}^{-n}_{\theta,\varphi}(\mathbb{D})\setminus\mathbb{D} \text{ for all } n\in\mathbb{N},\\ \sigma_0 & \text{on } \widehat{\mathbb{C}}\setminus\bigcup_{n=1}^\infty \mathfrak{B}^{-n}_{\theta,\varphi}(\mathbb{D}),\end{cases}$$

where σ_0 is the standard conformal structure on $\widehat{\mathbb{C}}$. The conformal structure $\sigma_{\theta,\varphi}$ is invariant under $\mathfrak{B}_{\theta,\varphi}$ and its maximal dilatation is the dilatation of H since H is quasiconformal and $B_{\theta,\varphi}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\Psi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $\Psi^*\sigma_0=\sigma_{\theta,\varphi}$. Therefore $\Psi\circ\mathfrak{B}_{\theta,\varphi}\circ\Psi^{-1}$ is a quadratic rational map. We normalize Ψ by $\Psi(0)=0$, $\Psi(1/\bar{b})=-1/\mu$ and $\Psi(\infty)=\infty$. Therefore we obtain that $F_{\lambda,\mu}=\Psi\circ\mathfrak{B}_{\theta,\varphi}\circ\Psi^{-1}$ since multipliers of fixed points are invariant under conjugation. The quadratic rational map $F_{\lambda,\mu}$ has a Siegel disk $\Delta=\Psi(\mathbb{D})$ with a critical point $\Psi(e^{2\pi i\varphi})\in\partial\Delta$. Moreover $\partial\Delta=\Psi(\mathbb{T})$ is a quasicircle since Ψ is quasiconformal. □

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