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On the Solutions of the Difference Equation $x_{n+1} = \frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2} x_{n-i}}$

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Abstract. We study the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2} x_{n-i}} \quad \text{for} \quad n = 0, 1, 2, ...$$

where $a, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0$ are the real numbers, $x_{-(2k+2)}x_{-(2k+1)}... x_0 \neq a$, and $k \in Z^+$.

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1. Introduction

Rational difference equations have been studied by several authors. Especially there has been a great interest in studying the attractivity, the solutions and the periodic nature of non-linear difference equations. In [1] Cinar has studied the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_n x_{n-1}}$ with the positive initial values. Also in [2] Cinar has studied the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$. In [3] Simsek has studied on the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$. In[4] Aloqeili has studied the solution of the difference equation $x_{n+1} = \frac{x_{n-1}}{a-x_{n-1}x_n}$, for $n = 1, 2, ...$and he has discussed the stability properties and semi-cycle behavior of this solution.

Our aim in this paper is to investigate the solutions of the difference equation

(1,1) $$x_{n+1}=\frac{ax_{n-(2k+2)}}{-a+\prod\limits_{i=0}^{2k+2}x_{n-i}} \quad \text{for} \quad n=0,1,2,...$$

where

$$a, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0 \text{ are the real numbers}$$

(1,2) $$x_{-(2k+2)}x_{-(2k+1)}...x_0 \neq a, a \neq 0 \text{ and } k \in Z^+.$$

Similar to the references in this paper, we define Eq.(1,1) with (1,2) and investigate the solutions of this difference equation.

2. Main Results

Theorem 1. *Assume that (1,2) holds and let* $\{x_n\}_{n=-(2k+2)}^{\infty}$ *be a solution of Eq.(1,1) with* $x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0$. *Then for* $n=0,1,2,...$ *all solutions of Eq.(1,1) are*

$$x_{(2k+3)n+1}=\begin{cases}\frac{ax_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}} & \text{, if } n \text{ is even}\\ x_{-(2k+2)} & \text{, if } n \text{ is odd}\end{cases} \quad ...(2,1)$$

$$x_{(2k+3)n+2}=\begin{cases}\left(\frac{1}{a}\right)x_{-(2k+1)}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right) & \text{, if } n \text{ is even}\\ x_{-(2k+1)} & \text{, if } n \text{ is odd}\end{cases} \quad ...(2,2)$$

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$$x_{(2k+3)n+2k+2}=\begin{cases}\left(\frac{1}{a}\right)x_{-1}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right) & \text{, if } n \text{ is even}\\ x_{-1} & \text{, if } n \text{ is odd}\end{cases} \quad ..(2,2k+2)$$

$$x_{(2k+3)n+2k+3}=\begin{cases}\frac{ax_0}{-a+x_0x_{-1}...x_{-(2k+2)}} & \text{, if } n \text{ is even}\\ x_0 & \text{, if } n \text{ is odd}\end{cases} \quad ...(2,2k+3)$$

Proof. $x_1, x_2, ..., x_{2k+3}$ are clear from Eq.(1,1). Also, for $n=1$ the result holds. Now suppose that $n>1$ and our assumption holds for $(n-1)$. We shall show that the result holds for n. From our assumption for $(n-1)$, we have

$$x_{(2k+3)n-(2k+2)}=\begin{cases}\frac{ax_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}} & \text{, if n is odd}\\ x_{-(2k+2)} & \text{, if n is even}\end{cases}$$

$$x_{(2k+3)n-(2k+1)}=\begin{cases}\left(\frac{1}{a}\right)x_{-(2k+1)}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right) & \text{, if n is odd}\\ x_{-(2k+1)} & \text{, if n is even}\end{cases}$$

$$\begin{array}{ll} . & . \\ . & . \\ . & . \end{array}$$

$$x_{(2k+3)n-1}=\begin{cases}\left(\frac{1}{a}\right)x_{-1}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right) & \text{, if n is odd}\\ x_{-1} & \text{, if n is even}\end{cases}$$

$$x_{(2k+3)n}=\begin{cases}\frac{ax_0}{-a+x_0x_{-1}...x_{-(2k+2)}} & \text{, if n is odd}\\ x_0 & \text{, if n is even}\end{cases}$$

Then, from Eq.(1,1) and above the equality, we have

$$x_{(2k+3)n+1}=\frac{ax_{(2k+3)n-(2k+2)}}{-a+\prod\limits_{i=0}^{2k+2}x_{(2k+3)n-i}}.$$

When n is odd, we have

$$\prod_{i=0}^{2k+2}x_{(2k+3)n-i}=\frac{ax_0x_{-1}...x_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}}.$$

Then

$$x_{2(k+3)n+1}=\frac{\frac{a^2x_{-(2k+2)}}{-a+x_0x_{-1}...x^n_{-(2k+2)}}}{-a+\frac{ax_0x_{-1}...x_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}}}=x_{-(2k+2)}.$$

When n is even, we have

$$\prod_{i=0}^{2k+2}x_{(2k+3)n-i}=x_0x_{-1}...x_{-(2k+2)}$$

and

$$x_{(2k+3)n+1}=\frac{ax_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}}.$$

Hence, we have

$$x_{(2k+3)n+1}=\begin{cases}\frac{ax_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}} & \text{, if n is even}\\ x_{-(2k+2)} & \text{, if n is odd}\end{cases}$$

Also, when n is even,

$$x_{(2k+3)n+2}=\frac{ax_{(2k+3)n-(2k+1)}}{-a+\prod\limits_{i=-1}^{2k+1}x_{(2k+3)n-i}}$$

and

$$\prod_{i=-1}^{2k+1}x_{(2k+3)n-i}=\frac{ax_0x_{-1}...x_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}}.$$

Then,

$$x_{(2k+3)n+2}=\frac{ax_{-(2k+1)}}{-a+\frac{ax_0x_{-1}...x_{-(2k+2)}}{-a+x_0x_{-1}...x_{-(2k+2)}}}$$

$$=\left(\frac{1}{a}\right)x_{-(2k+1)}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right).$$

When n is odd, we have

$x_{(2k+3)n+2} = \frac{a\left(\frac{1}{a}\right)x_{-(2k+1)}\left(-a+x_0x_{-1}...x_{-(2k+2)}\right)}{-a+x_0x_{-1}...x_{-(2k+2)}}$

and

$x_{(2k+3)n+2} = x_{-(2k+1)}$.

Hence, we have

$$x_{(2k+3)n+2} = \begin{cases} \left(\frac{1}{a}\right) x_{-(2k+1)} \left(-a + x_0 x_{-1}...x_{-(2k+2)}\right) & \text{, if n is even} \\ x_{-(2k+1)} & \text{, if n is odd} \end{cases}$$

Similarly, one can easily obtain (2,3),(2,4)... (2,2k+3). Thus, the proof is completed by indiction. ∎

Corollary 1. *Let $\{x_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(1,1) with Eq.(1.2). Assume that $a, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0 > 0$ and $x_{-(2k+2)}x_{-(2k+1)}...x_0 > a$. Then all solutions of Eq.(1,1) are positive.*

Proof. From the Eq.(2,1), (2,2), (2,3),..., (2,2k+3), all solutions of Eq.(1,1) are positive. ∎

Corollary 2. *Let $\{x_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(1,1) with Eq.(1.2). Assume that $a > 0, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0 < 0$. Then all solutions of Eq.(1,1) are positive when n is even. Otherwise all solutions are negative.*

Proof. It is obvious from the Eq.(2,1), (2,2), (2,3),..., (2,2k+3). ∎

Corollary 3. *Let $\{x_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(1,1) with Eq.(1.2). Assume that $a < 0, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0 < 0$ and $x_{-(2k+2)}x_{-(2k+1)}...x_0 > a$.Then all solutions of Eq.(1,1) are positive when n is even. Otherwise all solutions are negative.*

Proof. It is obvious from the Eq.(2,1), (2,2), (2,3),..., (2,2k+3). ∎

Corollary 4. *Let $\{x_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(1,1) with Eq.(1.2). Assume that $a < 0, x_{-(2k+2)}, x_{-(2k+1)}, ..., x_0 > 0$.Then all solutions of Eq.(1,1) are negative when n is even. Otherwise all solutions are positive*

Proof. It is obvious from the Eq.(2,1), (2,2), (2,3),..., (2,2k+3). ∎

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