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Nuclearity and Multipliers between Banach Spaces

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Abstract

Let ℓ be a Banach sequence space with a monotone norm $\|\cdot\|_\ell$, in which the canonical system (e_i) is a normalized symmetric basis. Let Λ be the class of such ℓ spaces and $\mathcal{D}(\ell,\tilde{\ell})$ be the space of all diagonal operators (that is multipliers) acting between $\ell,\tilde{\ell}\in\Lambda$. In [2], Djakov and Ramanujan considered the special case of multipliers on the class of Orlicz sequence spaces and proved that for Orlicz functions, if $\ell = l_M$ and $\tilde{\ell} = l_N$, then $\mathcal{D}(l_M, l_N) = l_{M_N^*}$, where $l_{M_N^*} := \sup\{N(st) - M(t) : t \in (0,1)\}$.

We consider the general form of multipliers on the class Λ and evaluate for some well known Banach sequence spaces. In Theorem 2.7, it is observed that quasidiagonal isomorphisms of different ℓ-Köthe spaces implies nuclearity which coincide with the common multipliers ($\Delta(\ell,\tilde{\ell}) := \mathcal{D}(\ell,\tilde{\ell})\cap\mathcal{D}(\tilde{\ell},\ell)$) of the corresponding spaces $\ell\in\Lambda$ and some results of [1] and [7] become the consequence of this theorem.

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1 Introduction

Let ℓ be a Banach sequence space in which $\{e_i = (\delta_{i,j})_{j\in\mathbb{N}} : i \in \mathbb{N}\}$ forms an unconditional basis. The norm $\|\cdot\|_\ell$ is called *monotone* [3] if $\|x\|_\ell \le \|y\|_\ell$ whenever $x = (\xi_i)$, $y = (\eta_i)$, $|\xi_i| \le |\eta_i|$, $i \in \mathbb{N}$. By Λ we denote the set of all such spaces ℓ with monotone norm and by $\Lambda^{(s)}$ the class of those of them with symmetric canonical basis $\{e_i\}$.

A matrix $A := (a_{i,n})_{i,n\in\mathbb{N}}$ of real numbers is called a *Köthe matrix* if $0 \le a_{i,n} \le a_{i,n+1}$ for each $i,n\in\mathbb{N}$; and for each $i\in\mathbb{N}$, there is $n\in\mathbb{N}$ such that $a_{i,n} > 0$.

Definition 1.1. *For* $\ell \in \Lambda$ *the* ℓ-Köthe space $K^{\ell}(A)$ (see [5, 6, 8]), defined by the Köthe matrix $A = (a_{i,n})_{i,n\in\mathbb{N}}$, is a Fréchet space of number sequences $\xi = (\xi_i)$ such that $(\xi_i a_{i,n}) \in \ell$, for each n, with the topology generated by the system of seminorms $\{|(\xi_i)|_n := \|(\xi_i a_{i,n})\| : n \in \mathbb{N}\}$.

Note that $|(e_i)|_n = |(e_i a_{i,n})| = a_{i,n}$. Hereafter the notation $e = (e_i)_{i\in\mathbb{N}}, e_i := (\delta_{i,k})_{k\in\mathbb{N}}$, will be always used for the canonical basis of $K^{\ell}(A)$ regardless of a matrix A. When ℓ is an l_p, we obtain the usual Köthe space

$K^{l_p}(A) = \{(\xi) = (\xi_i) : |(\xi_i)|_n^p = \sum_{i=1}^{\infty} |\xi_i|^p (a_{i,n})^p < \infty, \forall n \in \mathbb{N}\}$. Due to [11], it is known that every Fréchet space with an absolute basis is isomorphic to some l_1-Köthe space.

Set $\mathcal{P} := \{a = (a_i)_{i\in\mathbb{N}} : a_i \geq 1, \forall i\}$. For $a \in \mathcal{P}$ we introduce the weighted ℓ-space as $\ell(a) := \{x = (\xi_i) : \|x\|_{\ell(a)} := \|(\xi_i\, a_i)\|_{\ell} < \infty\}$. For a given sequences $a = (a_i)_{i\in\mathbb{N}} \in \mathcal{P}$ and $\lambda_n \to \alpha$, $-\infty < \alpha \leq \infty$, we call the ℓ-Köthe space $E_{\alpha}^{\ell}(a) := K^{\ell}(\exp(\lambda_n a_i))$, *$\ell$-power series space of finite (respectively, infinite) type* if $\alpha < \infty$ (respectively, $\alpha = \infty$).

Let $X = K^{\ell}(A)$ and $\tilde{X} = K^{\ell}(\tilde{A})$ be ℓ-Köthe spaces with the canonical bases (e_i). We say that X is quasidiagonally isomorphic to $\tilde{X}$ and write $X \overset{qd}{\simeq} \tilde{X}$ if there exists $T : X \to \tilde{X}$ such that $Te_i := t_i e_{\varphi(i)},\quad i \in \mathbb{N}$, is an isomorphism, where t_i is a sequence of numbers and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection.

A function $M : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is convex and $M(0) = 0$; $\lim_{t\to\infty} M(t) = \infty$. A function M is said to be *degenerate*, if it vanishes for some $t > 0$. It is clear that any Orlicz function is continuous, non-decreasing and satisfies the condition :$CM(t) \leq M(Ct),\ C > 1$.

Each Orlicz function M determines the Banach space l_M, so called *Orlicz space* of all scalar sequences $x = (\xi_k)$ such that $\sum_{k=1}^{\infty} M(|\xi_k|/r) < \infty$ for some $r > 0$, equipped with the norm:$\|x\| = \|x\|_{l_M} := \inf\{r > 0 : \sum_{k=1}^{\infty} M(|\xi_k|/r) \leq 1\}$.

We introduce an order relation on the class of all Orlicz functions by setting $M_1 \preceq M_2$ if there are constants $C > 0, \delta > 0$ such that $M_1(t) \leq M_2(Ct)$ for $t \in [0, \delta]$.

Remark: The relation $\preceq$ is a proper partial order. Since for any Orlicz function M there exists an Orlicz function N which is not comparable with M (see, e.g. [9]). We refer to [10] for the basic theory of Orlicz functions and Orlicz sequence space.

In [1], it is proved that, for different Köthe matrices A and B, quasidiagonal isomorphisms of l_p-Köthe space and l_q-Köthe space $(1 \leq p \neq q < \infty)$ implies "nuclearity" of the spaces (see 2.4). In [7], quasidiagonal isomorphisms of l_M-Köthe spaces and l_N-Köthe spaces for different Orlicz functions M and N implies some kind of the "nuclearity" of the spaces (see 2.5). These two result are generalized in Theorem 2.7, and it is observed that quasidiagonal isomorphisms of different ℓ-Köthe spaces implies nuclearity which coincide with the multipliers of the corresponding spaces $\ell \in \Lambda$ and the results in [1] and [7] become the consequence of this theorem.

2 Multipliers

Let $\mathcal{D}(\ell, \tilde{\ell})$ be the space of all diagonal operators (that is multipliers) acting between $\ell, \tilde{\ell} \in \Lambda$. Set $\Delta(\ell, \tilde{\ell}) := \mathcal{D}(\ell, \tilde{\ell}) \cap \mathcal{D}(\tilde{\ell}, \ell)$. One can easily observe the following equalities :

(A) For any $p, q \geq 1$, $\mathcal{D}(l_p, l_q) = \begin{cases} l_r & \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \quad \text{if} \quad p > q, \\ l_\infty & \text{if} \quad p \leq q, \end{cases}$

(B) $\mathcal{D}(l_1, \ell) = l_\infty$ for each $\ell \in \Lambda$,

(C) $\mathcal{D}(l_\infty, \ell) = \ell$ for each $\ell \in \Lambda$,

(D) $\mathcal{D}(\ell, l_\infty) = l_\infty$ for each $\ell \in \Lambda$,

(E) $\mathcal{D}(\ell, l_1) = \ell^*$ for each $\ell \in \Lambda$, where ℓ^* is dual of $\ell \in \Lambda$.

(F) $\mathcal{D}(\ell, \ell) = l_\infty$ for each $\ell \in \Lambda$; in particular $\mathcal{D}(l_\infty, l_\infty) = l_\infty$,

(G) $\mathcal{D}(\ell, \tilde{\ell}) \in \Lambda$,

(H) $\mathcal{D}(l_M, l_N) = l_\infty$ if and only if $M \preceq N$

(I) $\mathcal{D}(\tilde{\ell}, \ell) = \mathcal{D}(\ell, \tilde{\ell}) = l_\infty$ then $\tilde{\ell} = \ell$.

(J) $\ell \in \Lambda^{(s)}$ if and only if $\mathcal{D}(\ell, \ell_\sigma) = l_\infty$ for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$

It is convenient to realize a Cartesian product of $\ell \in \Lambda$ and $\tilde{\ell} \in \Lambda$, as a Banach sequence space $\underline{\ell \times \ell}$ such that $(x, y) \to z = (\zeta_i)$ and $\zeta_i = \begin{cases} \xi_k & \text{if} \quad i = 2k-1, \\ \eta_k & \text{if} \quad i = 2k, \end{cases}$ where $x = (\xi_k) \in \ell$ and $y = (\eta_k) \in \tilde{\ell}$.

For $1 \leq p < q < \infty$, we easily obtain that

1. $\mathcal{D}(\underline{l_q \times l_p}, \underline{l_p \times l_q}) = \underline{l_\infty \times l_r}$,
2. $\mathcal{D}(\underline{l_p \times l_q}, \underline{l_q \times l_p}) = \underline{l_r \times l_\infty}$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Theorem 2.1. *For $\ell, \tilde{\ell} \in \Lambda$ the following are equivalent:*

(i) $\ell \neq \tilde{\ell}$

(ii) *One of the $\mathcal{D}(\tilde{\ell}, \ell)$, $\mathcal{D}(\ell, \tilde{\ell})$ is non-trivial (that is, not equal to l_∞.)*

Proof. Regarding (F) and (I), $\ell = \tilde{\ell}$ if and only if $\mathcal{D}(\ell, \tilde{\ell}) = l_\infty = \mathcal{D}(\tilde{\ell}, \ell)$. □

Theorem 2.2. *If $\ell \neq \tilde{\ell}$ then*

1. *there exists a Köthe matrix A such that $K^{\ell}(A) \neq K^{\tilde{\ell}}(A)$*
2. *there exists a positive sequence $a = (a_i)$ such that $E^{\ell}_{\nu}(a) \neq E^{\tilde{\ell}}_{\nu}(a)$, $\nu = 0, \infty$.*

Proof. We consider the case (i), since the case (ii) can be obtained analogously. If $\ell \neq \tilde{\ell}$ then by the Theorem 2.1, one of the $\mathcal{D}(\tilde{\ell}, \ell)$, $\mathcal{D}(\ell, \tilde{\ell})$ is non-trivial. Without loss of generality, say $\mathcal{D}(\ell, \tilde{\ell}) \neq l_\infty$. Thus for each $x = (\xi_i) \in \ell$ there exist $\lambda = (\lambda_i) \in \mathcal{D}(\ell, \tilde{\ell})$, such that $(\lambda_i \xi_i) \in \tilde{\ell}$.

Suppose $(a_{ip}) := (\lambda_i)$ for each p. Thus, $(\xi_i a_{ip}) \in \ell$ and also $(\xi_i a_{ip}) \in \tilde{\ell}$ which implies that $K^{\ell}(A) \subset K^{\tilde{\ell}}(A)$. But $K^{\ell}(A) \not\supset K^{\tilde{\ell}}(A)$. Indeed, if not, $(\xi_i) \in K^{\tilde{\ell}}(A)$ and $(\xi_i a_{ip}) \in \tilde{\ell}$ which implies that $(\xi_i a_{ip}) \in \ell$. Thus, $(a_{ip}) \in \mathcal{D}(\tilde{\ell}, \ell)$ which contradicts the assumption $\mathcal{D}(\tilde{\ell}, \ell) = l_\infty$. □

Nuclearity of the classical Köthe spaces $K^{l_p}(A)$ is equivalent to the Grothendieck-Pietsch condition [13]; $\forall p \ \exists q > p$ and $(\xi_n) \in l_1$ with $a_n^p \leq \xi_n a_n^q$, $n \in N$. In [14], it is proved that $K^{\ell}(A)$ is nuclear if and only if A satisfies the Grothendieck-Pietsch condition.

Lemma 2.3. *$K^{l_p}(a_{nk})$ is nuclear if and only if*

$$\exists r \forall k \exists \sigma(k) \ : \ \sum_{i=1}^{\infty} \left(\frac{a_{nk}}{a_{n\sigma(k)}} \right)^r < \infty \tag{1}$$

where $\sigma : \mathbb{N} :\rightarrow \mathbb{N}$ is an injection map.

Proof. It is well known fact that (see [12]) $K^p(a_{nk})$ is nuclear if and only if

$$\sum_{i=1}^{\infty} \frac{a_{nk}}{a_{n\phi(k)}} < \infty. \tag{2}$$

So it is sufficient to show that (2.3) implies (2). Assume that (2.3) holds. Fix $r \in \mathbb{N}$, for any $k \in \mathbb{N}$ there exists an injective map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{i=1}^{\infty} \left(\frac{a_{nk}}{a_{n\phi(k)}} \right)^r < \infty$, that is, $\left(\frac{a_{nk}}{a_{n\phi(k)}} \right) \in l_r$. In the same way, for $\phi(k)$, there exist $\phi^2(k)$ such that $\sum_{i=1}^{\infty} \left(\frac{a_{n\phi(k)}}{a_{n\phi^2(k)}} \right)^r < \infty$, that is, $\left(\frac{a_{n\phi(k)}}{a_{n\phi^2(k)}} \right) \in l_r$.

If we determine recursively $\phi^s(k)$, for $s = 1, ..., r$ so that $\sum_{i=1}^{\infty} \left(\frac{a_{n\phi^{s-1}(k)}}{a_{n\phi^s(k)}} \right)^r < \infty$, that is, $\left(\frac{a_{n\phi^{s-1}(k)}}{a_{n\phi^s(k)}} \right) \in l_r$.

So we have, $\left(\frac{a_{nk}}{a_{n\phi^r(k)}}\right) = \left(\frac{a_{nk}}{a_{n\phi(k)}}\right).\left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right)...\left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right)$

By generalized Hölder's inequality

$$\sum_{n=1}^{\infty}\left|\left(\frac{a_{nk}}{a_{n\phi^r(k)}}\right)\right| = \sum_{n=1}^{\infty}\left|\left(\frac{a_{nk}}{a_{n\phi(k)}}\right).\left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right)...\left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right)\right|$$
$$\leq C\left\|\left(\frac{a_{nk}}{a_{n\phi(k)}}\right)\right\|_{l_r}.\left\|\left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right)\right\|_{l_r}...\left\|\left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right)\right\|_{l_r}$$

Since all factors are finite, product is finite. □

Definition 2.4. *$A \in \mathbb{N}(\ell, \tilde{\ell})$ if for each p, there exists q such that $(\frac{a_{ip}}{a_{iq}}) \in \Delta(\ell, \tilde{\ell})$.*

Example 2.5. *(see[1], Prop.4) If $K^{l_p}(A) \overset{qd}{\simeq} K^{l_q}(B)$ then $K^{l_p}(A)$ is nuclear. Indeed $\Delta(l_p, l_q) = l_r$ and regarding the Lemma 2.3, $A \in \mathbb{N}(l_p, l_q)$*

Example 2.6. *(see[7]) If $K^{l_M}(A) \overset{qd}{\simeq} K^{l_N}(B)$ then $K^{l_M}(A)$ is nuclear. Indeed, $\Delta(l_M, l_N) = l_{M_N^*}$ and $A \in \mathbb{N}(l_M, l_N)$*

Theorem 2.7. *For $\ell, \tilde{\ell} \in \Lambda$, if $K^{\ell}(A) = K^{\tilde{\ell}}(A)$, then $A \in \mathbb{N}(\ell, \tilde{\ell})$.*

Proof. Let $a_p := a_{ip}$. For each p, there exists q such that

$$\tilde{\ell}(a_p) \subset \ell(a_q) \text{ and } \ell(a_p) \subset \tilde{\ell}(a_q) \tag{3}$$

Take $x = (\xi_i) \in \ell$. Then $(\frac{\xi_i}{a_{iq}}) \in \ell(a_q)$. Due to 3 we have $(\frac{\xi_i}{a_{iq}}) \in \tilde{\ell}(a_p)$ which implies that $(\xi_i\frac{a_{ip}}{a_{iq}}) \in \tilde{\ell}$. Thus $(\frac{a_{ip}}{a_{iq}}) \in \mathcal{D}(\ell, \tilde{\ell})$. Analogously, we obtain that $(\frac{a_{ip}}{a_{iq}}) \in \mathcal{D}(\tilde{\ell}, \ell)$ which implies that $(\frac{a_{ip}}{a_{iq}}) \in \Delta(\ell, \tilde{\ell})$. □

Corollary 2.8. *$K^{\ell}(A) = K^{\ell_\sigma}(A)$ if and only if $\mathbb{N}(\ell, \ell_\sigma)$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is injection.*

For $A = (a_{ip})$, if for each p, there exist $q, C > 0$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is injection such that $a_{ip} \leq Ca_{\sigma(i)q}$ and $a_{\sigma(i)p} \leq Ca_{iq}$ then we say that the matrix (a_{ip}) is σ-equivalent and denote $(a_{ip}) \equiv a_{\sigma(i)p}$. Let $\mathcal{G}(A) := \{\sigma : \mathbb{N} \to \mathbb{N} \text{ injection} : (a_{ip}) \equiv a_{\sigma(i)p}\}$

Theorem 2.9. *$K^{\ell}(A) \overset{qd}{\simeq} K^{\ell_\sigma}(A)$ if and only if there exists $\gamma \in \mathcal{G}(A)$ such that $A \in \mathbb{N}(\ell, \ell_{\sigma\circ\gamma})$*

Note that $K^{\ell}(A) \overset{qd}{\simeq} K^{\ell_\sigma}(A)$ if and only if $K^{\ell}(A) \overset{qd}{\simeq} K^{\ell_{\sigma\circ\gamma}}(A)$

Problem 2.10. *For Tsirelson space $\mathbb{T}$ (see [4]), we observe $\mathcal{D}(l_2, \mathbb{T}) = l_\infty$ We claim that $l_p \subset \Delta(l_2, \mathbb{T}) \subset l_\infty$ and also $\mathcal{D}(\mathbb{T}, l_2) \neq l_\infty$ but what exactly it is?*

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