Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 30, 1493 - 1504

Applying Differential Transformation Method to the One-Dimensional Planar Bratu Problem

I. H. Abdel-Halim Hassan

Department of Mathematics, Faculty of Science,
Zagazig University, Zagazig, Egypt
ismhalim@hotmail.com

Vedat Suat Ertürk

Department of Mathematics, Faculty of Arts and Sciences
Ondokuz Mayis University, Samsun,Turkey
vserturk@omu.edu.tr

Abstract

This paper is the application of differential transformation method (DTM) to solve the Bratu problem**.** A considerable research works have been conducted recently in applying DTM to different types of partial differential equation of Abdel-Halim Hassan (Chaos, Solitons & Fractals [6]) and fractional differential equations of Arıkoğlu and Özkol (Chaos, Solitons & Fractals [1]). The nonlinear eigenvalue problem $\Delta u + \lambda\, e^{u} = 0$ in the unit square with u = 0 on the boundary is often referred to as "the Bratu problem". The Bratu problem in one-dimensional planar coordinates $u'' + \lambda e^{u} = 0$ with u(0) = u(1) = 0 has two known, bifurcated solutions for values of $\lambda < \lambda_c$, no solutions for $\lambda > \lambda_c$ and a unique solution when $\lambda = \lambda_c$. The value of λ_c is related to the fixed point of hyperbolic cotangent function. Two special cases of the problem are illustrated by using the technique and numerical results and conclusions will be presented.

Mathematics Subject Classification: 74S30, 34B15

Keywords: Differential transformation method, Bratu problem

1. Introduction

In this letter, the results of applying differential transformation method [9] to the one-dimensional planar Bratu problem [5] will be presented. The Bratu problem is an elliptic partial differential equation which comes from a simplification of the solid fuel ignition model in thermal combustion theory [4]. It is also a nonlinear eigenvalue problem that is often used as a benchmarking tool for numerical methods [14], [10], [11] , [8] and [4]. The goal of this paper will be to apply differential transformation method to the 1- dimensional planar Bratu problem and compare the known exact solutions to numerical solutions produced by differential transformation method.

The classical Bratu problem is

$$\Delta u + \lambda e^{u} = 0 \text{ on } \Omega : \{(x, y) \in 0 \le x \le 1, 0 \le y \le 1\}, \tag{1.1}$$

$$\text{with } u = 0 \text{ on } \partial\Omega . \tag{1.2}$$

The one-dimensional (planar) version of this problem is

$$u'' + \lambda\, e^{u} = 0 \qquad 0 \le x \le 1, \tag{1.3}$$

$$\text{with } u(0) = 0 \text{ and } u(1) = 0. \tag{1.4}$$

We will organize our paper in the following way. In Section 2, the differential transformation method will be presented. In Section 3, the differential transformation method will be applied to the Bratu problem. Section 4, the algorithm is implemented for two numerical examples. Conclusion will be presented in the last section.

2. Description of differential transformation method

In order to solve the Bratu problem by differential transformation its basic theory is stated in brief in this section. The differential transformation of the *k*th derivative of a function $u(x)$ is defined as follows:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}. \tag{2.1}$$

In (2.1), $u(x)$ is the original function and $U(k)$ is the transformed function.

As in [3] and [7] the differential inverse transformation of $U(k)$ is defined as follows:

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k. \tag{2.2}$$

In fact, from (2.1) and (2.2), we obtain

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x - x_0)^k \left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}. \tag{2.3}$$

Eq.(2.3) implies that the concept of differential transformation is derived from the Taylor series expansion. From the definitions of (2.1) and (2.2), it is easy to prove that the transformed functions comply with the following basic mathematics operations(see Table 1, below).

Table 1. The fundamental mathematics operations

Original function	Transformed function	
$z(x) = u(x) \pm v(x)$	$Z(k) = U(k) + V(k)$	(2.4)
$z(x) = \lambda\, u(x)$	$Z(k) = \lambda\, U(k)$	(2.5)
$z(x) = du(x)/dx$	$Z(k) = (k+1)\, U(k+1)$	(2.6)
$z(x) = d^2u(x)/dx^2$	$Z(k) = (k+1)\,(k+2)U(k+2)$	(2.7)
$z(x) = u(x)v(x)$	$Z(k) = \sum_{l=0}^{k} U(l)V(k-l)$	(2.8)
$z(x) = \lambda\, x^m$	$Z(k) = \lambda\delta(k-m), \delta(k-m) = \begin{cases} 1, & if\ \ k = m, \\ 0, & if\ \ k \neq m. \end{cases}$	(2.9)

In real applications, the function $u(x)$ is expressed by a finite series and (2.2) can be written as

$$u(x) = \sum_{k=0}^{n} U(k)(x - x_0)^k \,. \tag{2.10}$$

Equation (2.10) implies that $\sum_{k=n+1}^{\infty} U(k)(x - x_0)^k$ is negligibly small.

3. Numerical differential transformation algorithm

In this section, the differential transformation method is implemented for the solution of the Bratu problem which is given by

$$u'' + \lambda\, e^u = 0 \tag{3.1}$$

$$u(0) = 0 \text{ and } u(1) = 0, \tag{3.2}$$

where $\lambda > 0$. The exact solution of (3.1) and (3.2) is given [14], [11] and [12] presented here as

$$u(x) = -2\ln\left[\frac{\cosh\left((x-\frac{1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \tag{3.3}$$

where θ solves

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{3.4}$$

There are two solutions to (3.4) for values of $0 < \lambda < \lambda_c$. For $\lambda > \lambda_c$,there are no solutions. The solution (3.3) is only unique for a critical value of $\lambda = \lambda_c$ which solves

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right). \tag{3.5}$$

It was evaluated in [14], [11], [12] , [13] , [8] and [4] , that the critical value λ_c is given by

$$\lambda_c = 3.513830719. \tag{3.6}$$

The technique consists first of taking differential transformation to both sides of Eq. (3.1).Before doing this, let's expand the non-linear term e^u as follows:

$$e^u = \sum_{n=0}^{\infty}\frac{u^n}{n!}. \tag{3.7}$$

Theorem 1. If $z(x) = \sum_{n=0}^{\infty}\left[\frac{(u(x))^n}{n!}\right]$ then

$$Z(k) = \sum_{n=0}^{r}\frac{1}{n!}\sum_{k=0}^{\infty}\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k.$$

Proof. By using Eq.(2.8), we obtain differential transformation of $(u(x))^n$ as

$$(u(x))^n = \sum_{k=0}^{\infty} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k.$$

From hypothesis, and by using Eq.(2.5) we obtain

$$Z(k) = \sum_{n=0}^{r} \frac{1}{n!} \sum_{k=0}^{\infty} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k.$$

Theorem 2. If $z(x)=\exp(u(x))$ then

$$Z(k) = \sum_{k=0}^{\infty} \sum_{n=0}^{r} \frac{1}{n!} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k.$$

Proof. By using Theorem 1, we find

$$\exp(u(x)) = \sum_{n=0}^{r} \frac{1}{n!} \sum_{k=0}^{\infty} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k$$

or

$$Z(k) = \sum_{k=0}^{\infty} \sum_{n=0}^{r} \frac{1}{n!} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times$$
$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-x_0)^k.$$

Finally if we apply differential transformation to (3.1)

$$U(k+2) = -\frac{\lambda}{(k+1)(k+2)} \left(\begin{array}{l} \displaystyle\sum_{k=0}^{\infty} \sum_{n=0}^{r} \frac{1}{n!} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)\ldots\times \\ U(k_{n-1}-k_{n-2})U(k-k_{n-1})x^k \end{array} \right). \tag{3.8}$$

The boundary conditions in Eq.(3.2) can be transformed at $x_0=0$ as follows:

$$U(0) = 0 \tag{3.9}$$

and

$$\sum_{k=0}^{n} U(k) = 0. \tag{3.10}$$

Let

$$U(1) = a\,. \tag{3.11}$$

Substituting (3.9) and (3.11), and $k = 0$ into (3.8), we have

$$U(2) = -\frac{\lambda}{2}\,. \tag{3.12}$$

Substituting (3.9),(3.11)-(3.12), and $k = 1$ into (3.8), we have

$$U(3) = -\frac{a\lambda}{6}. \tag{3.13}$$

Following the same recursive procedure, we calculate up to the *n*th term $U(n)$.

The solution obtained from Eq. (2.10) has yet to satisfy the second initial condition in Eq.(1.4), which has not been manipulated in obtaining this approximate solution. Applying this initial condition and then solving the resulting equation for *a*, will determine the unknown constant *a* and eventually the numerical solution.

4. Numerical examples

For purposes of illustration of differential transformation method for solving the one-dimensional planar Bratu problem, The computer application program "Mathematica" was used to execute the algorithm that was used with the numerical examples. The differential transformation method described in Section 2 is applied to two special cases of the Bratu problem. The first one is when $\lambda = 1$. In this case, the problem (3.1)-(3.2) has two locally unique solutions $u_1(x)$ and $u_2(x)$ with $u_1'(0) \approx 0.549$ and $u_2'(0) \approx 10.909$ (see[14]). The solution of the differential transformation method converges to the solution $u_1'(0) \approx 0.549$ and not to the solution $u_2'(0) \approx 10.909$. In Table 2, the exact solution for the case $\lambda = 1$ derived from Eq. (3.3) is compared with the numerical solution obtained by the differential transformation technique. Table 1 shows that the absolute errors, i.e. the difference between the approximate solution and exact solution. Fig. 1 shows the analytic solutions and numerical solutions obtained by differential transformation method for $\lambda = 1$.

Table 2. Differential transformation method approximation for the Bratu problem for the case $\lambda = 1$ and $r = 6$

x	Exact solution	Numerical solution	Error
0.1	0.0498467912	0.0498467912	0.0
0.2	0.0891899346	0.0891899345	0.0000000001
0.3	0.1176090958	0.1176090956	0.0000000002
0.4	0.1347902539	0.1347902537	0.0000000002
0.5	0.1405392144	0.1405392142	0.0000000002
0.6	0.1347902539	0.1347902537	0.0000000002
0.7	0.1176090958	0.1176090955	0.0000000003
0.8	0.0891899346	0.0891899344	0.0000000002
0.9	0.0498467912	0.0498467910	0.0000000002
1.0	0.0	0.0	0.0

Table 3. Differential transformation method approximation for the Bratu problem for the case $\lambda = 2$ and $r = 6$

x	Exact solution	Numerical solution	Error
0.1	0.1144107433	0.1144120397	0.0000012964
0.2	0.2064191165	0.2064216798	0.0000025633
0.3	0.2738793118	0.2738830790	0.0000037672
0.4	0.3150893642	0.3150942368	0.0000048726
0.5	0.3289524213	0.3289582675	0.0000058462
0.6	0.3150893642	0.3150960238	0.0000066596
0.7	0.2738793118	0.2738866035	0.0000072917
0.8	0.2064191165	0.2064268222	0.0000077057
0.9	0.1144107433	0.1144181533	0.0000074100
1.0	0.0	0.0	0.0

The second case is when $\lambda = 2$. Table 3 shows that the absolute error of the approximation of the numerical technique. As λ approaches the critical value λ_c, the error becomes larger , and the convergence becomes slower as is clear for the case $\lambda = 2$ shown in Table 3. Also, Fig 2 shows the analytic solutions and numerical solutions obtained by differential transformation method for $\lambda = 2$.

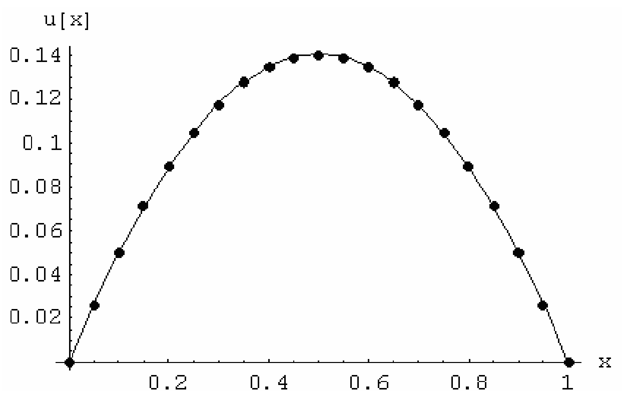

Fig.1. The differential transformation solution (dotted curve) versus analytic solution (solid curve) for $\lambda = 1$.

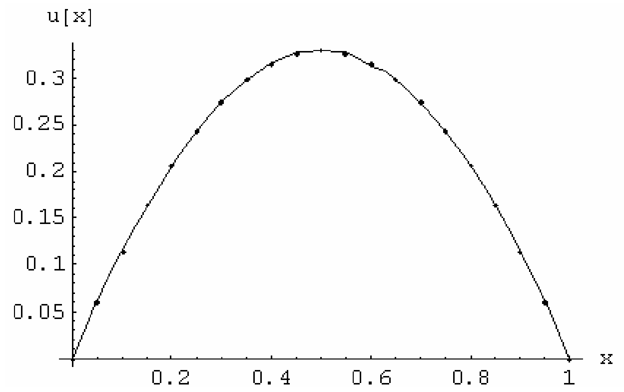

Fig. 2. The differential transformation solution (dotted curve) versus analytic solution (solid curve) for $\lambda = 2$.

5. Conclusion

Differential transformation method has been applied to the Bratu problem. The results for two numerical examples showed that the present method is quite reliable. Therefore, this method can be applied to many complicated non-linear eigenvalue problem and does not require linearization, discretization or perturbation.

References

[1] A. Arıkoğlu , İ. Özkol, Solution of fractional differential equations by using differential transform method, Chaos, Solitons and Fractals, 34 (2007) ,1473–1481.

[2] A. M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu-type equations, Applied Mathematics and Computation, 166 (2005), 652-663.

[3] C.K. Chen, S.H. Ho, Application of differential transformation to eigenvalue problems, Applied Mathematics and Computation ,79 (1996),173-188.

[4] D.A. Frank-Kamenetski, Diffusion and Heat Exchange in Chemical Kinetics, Princeton University Press: Princeton, NJ. 1955.

[5] G. Bratu, Sur les équations intégrales nonlinéaires, Bulletins of the Mathematical Society of France, 42 (1914), 113-142.

[6] I.H. Abdel-Halim Hassan, Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems, Chaos, Solitons and Fractals (2006), doi:10.1016/j.chaos.2006.06.040.

[7] I.H. Abdel-Halim Hassan, Different applications for the differential transformation in the differential equations, Applied Mathematics and Computation,129 (2002),183-201.

[8] J. Jacobson, K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, Journal of Differentail Equations,184 (2002), 283-298.

[9] J. K. Zhou, Differential Transformation and Its Applications for Electrical Circuits (in Chinese), Huazhong Univ.Press, Wuhan, China, 1986.

[10] J.P. Abbott, An efficient algorithm for the determination of certain bifurcation points, Journal of Computational and Applied Mathematics, 4 (1978),19-27.

[11] J.P. Boyd, Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one-dimensional Bratu equation, Applied Mathematics and Computation, 142 (2003), 189-200.

[12] J.P. Boyd, An analytical and numerical study of the two-dimensional Bratu equation, Journal of Scientific Computing, 1 (1986), 183-206.

[13] R. Buckmire, Investigations of nonstandard Mickens-type finite-differences schemes for singular boundary value problems in cylindrical and spherical coordinates, Numerical Methods for Partial Differential Equations 19 (2003), 380-398.

[14] U.M. Ascher, R.M.M. Matheij and R.D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, Society for Industrial and Applied Mathematics: Philadelphia, PA. 1995.

Received: June 3, 2007