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k-Kernels in Orientations of the Line Graph

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Abstract

Let G be a graph, $L(G)$ its line graph and D an orientation of $L(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle has two symmetrical arcs.

In this paper we prove: If every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has a chord (i, j) such that at least one of the two following properties holds:

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) with $(r, s) \neq (j, i)$,

then D has a k-kernel, ($k \geq 3$).

Mathematics Subject Classification: 05C20

Keywords: kernel, k-kernel, line graph

1 Introduction

For general concepts we refer the reader to [1]. Let G be a graph; $V(G)$ and $E(G)$ will denote the sets of vertices and edges of G respectively. The line graph $L(G)$ of G is defined as follows: $V(L(G)) = E(G)$; for $a, b \in E(G)$, a is adjacent to b in $L(G)$ if and only if they are adjacent as edges in G (i.e. they have a common endpoint). A digraph D is an orientation of G if D is obtained by directing each edge of G in at least one of the two possible directions. If $S \subseteq V(G)$ or $T \subseteq E(G)$, then $G[S]$ and $G[T]$ will denote the subgraphs of G induced by S and T respectively.

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. symmetrical part of D) which is denoted by Asym(D) (resp. Sym(D)) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

If $\mathcal{C}$ is a walk of G (resp. a directed walk of D) we will denote by $\ell(\mathcal{C})$ its length.

Along this work all notation will be taken modulo n without more explanation.

A cycle of G (resp. a directed cycle of D) is a sequence of vertices of G (resp. of D), $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ such that $[i, i+1] \in E(G)$ (resp. $(i, i+1) \in A(D)$), for $i \in \{0, 1, \ldots, n-1\}$.

Walks, paths and cycles we mean as partial subgraphs or partial subdigraphs.

Let $\mathcal{C}$ be a cycle of G (resp. a directed cycle of D). For $\{i.j\} \subseteq V(\mathcal{C})$ we denote by $[i, \mathcal{C}, j]$ (resp. by $(i, \mathcal{C}, j)$) the path from i to j, $[i, i+1, i+2, \ldots, j]$ (resp. the directed path $(i, i+1, i+2, \ldots, j)$) contained in $\mathcal{C}$. A chord of $\mathcal{C}$ is an edge (resp. an arc) $[i, j] \in A(G) - A(\mathcal{C})$ (resp. $(i, j) \in A(D) - A(\mathcal{C})$) such that $1 < \ell(i, \mathcal{C}, j) < \ell(\mathcal{C}) - 1$; with $\{i, i+1, \ldots, j\} \subseteq V(\mathcal{C})$.

By the directed distance $d_D(x, y)$ from the vertex x to vertex y in a digraph D we mean the length of the shortest directed path from x to y in D. We put $d_D(x, y) = \infty$ if there is no directed path from x to y in D.

Let k be a natural number with $k \geq 2$. A set $J \subseteq V(D)$ will be called a k-kernel of the digraph D if:

(1) for $\{x, x'\} \subseteq J$ we have $d_D(x, x') \geq k$ and

(2) for each $y \in (V(D) - J)$ there exists $x \in J$ such that $d_D(y, x) \leq k - 1$.

k-kernels were first defined and considered by M. Kwaśnik in [6]. In [6] M. Kwaśnik also proved the following interesting result: Let D be a strongly connected digraph such that every directed cycle of D has length $\equiv 0 (\operatorname{mod} k), k \geq 2$. Then D has a k-kernel.

For $k = 2$ we have a kernel in the sense of Berge [1]. When every induced subdigraph of D has a kernel, D is said to be kernel-perfect or a KP-digraph.

In 1976 H. Meyniel [3] conjectured: Let D be a digraph; if every odd directed cycle of D possesses two chords, then D is a KP-digraph.

In general the condition that each odd directed cycle has two chords is not sufficient for a digraph to be kernel-perfect. In [4] Galeana-Sánchez constructed for each k a triangle free digraph D_k with no kernel such that every odd directed cycle in D_k has at least k chords. Still under some restrictions on the structure of the underlying unoriented graph of a digraph D the condition: Each odd directed cycle has two chords is not enough for a digraph to be kernel-perfect. However in [2] O. V. Borodin, A. V. Kostochka and D. R. Woodall proved: Let H be the line graph of a graph G; an orientation D of H is kernel-perfect if and only if each odd directed cycle has a chord and each clique has a kernel.

A feasible extention of the Meyniel's Conjecture for k-kernels $k \geq 2$, would say: Let D be a digraph, if every directed cycle of length$\not\equiv 0 \pmod k$ has two chords, then D has a k-kernel.

In this paper we prove that this assertion is not true for digraphs in general; also we prove the following extention for k-kernels ($k \geq 3$) of the result of Borodin, Kostochka and Woodall: Let G be a graph, $L(G)$ its line graph and D an orientation of $L(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle has two symmetrical arcs. If every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has a chord (i, j) such that at least one of the two following properties holds:

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) with $(r, s) \neq (j, i)$,

then D has a k-kernel, ($k \geq 3$).

As a consequence it is proved the following assertion which is a particular case in which the feasible extention of the Meyniel's Conjecture for $k \geq 3$; holds: Let G be a graph and D an orientation of $L(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle is symmetrical. If every directed cycle of D whose length is $\not\equiv 0 (\bmod k)$ has two chords, then D has a k-kernel, $k \geq 3$.

This two results also are generalizations of the Kwaśnik's result for special orientations of the line graph.

2 k-kernels in orientations of the line graph

In this section we prove the main result of this paper which is ennounced in the abstract, before of this, we prove some Lemmas which show structural properties of the line graph.

Lemma 2.1 *Let G be a graph, $L(G)$ its line graph and $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $L(G)$. If $[i, j] \in E(L(G)) - E(\mathcal{C})$ with $j \notin \{i-2, i+2\}$, then at least one of the following conditions holds:*

(a) $\{[s-1, s+1], [s, t]\} \subseteq E(L(G))$ *with; ($s = i$ and $t \in \{j-1, j+1\}$) or ($s = j$ and $t \in \{i-1, i+1\}$).*

(b) $\{[i-1, i+1], [j-1, j+1]\} \subseteq E(L(G))$.

(c) $L(G)[\{s-1, s, t, t+1\}] \cong K_4$ *with $s \in \{i, i+1\}$, $t \in \{j-1, j\}$.*

Proof: Let $i = [u, v] \in E(G)$ be given. We consider several possible cases:

Case 1 $i-1$ incides in u and $i+1$ incides in u.

Clearly in this case we have $[i-1, i+1] \in E(L(G))$.

If $j-1$ (resp. $j+1$) incides in some endpoint of i (u or v), then $[i, j-1] \in E(L(G))$ (resp. $[i, j+1] \in E(L(G))$) and (a) holds with $s = i$ and $t = j-1$ (resp. $s = i$ and $t = j+1$). So we can assume $j-1$ is not adjacent to i and $j+1$ is not adjacent to i. Let w be the endpoint of j such that $w \notin \{u, v\}$. Since $j-1$ (resp. $j+1$) is not adjacent to i but $j-1$ (and $j+1$) is adajcent to j it follows that $j-1$ (and $j+1$) incides in w; so $[j-1, j+1] \in E(L(G))$ and (b) holds.

Case 2 $i-1$ incides in v and $i+1$ incides in v.

In this case the proof is exactly as those of Case 1.

Case 3 $i-1$ incides in u and $i+1$ incides in v.

Since j is adjacent to i, we have that j incides in u or j incides in v.

First suppose that j incides in u.

If $j-1$ (resp. $j+1$) incides in u, then (c) holds with $s = i$ and $t = j-1$ (resp. $s = i$ and $t = j$). So we can assume that both $j-1$ and $j+1$ incide in the other endpoint of j and then $[j-1, j+1] \in E(L(G))$ and (a) holds with $s = j$ and $t = i-1$. Now suppose that j incides in v.

When $j-1$ (resp. $j+1$) incides in v then (c) holds with $s = i+1$ and $t = j-1$ (resp. $s = i+1$ and $t = j$). When $j-1$ and $j+1$ both incide in the other endpoint of j we obtain $[j-1, j+1] \in E(L(G))$ and (a) holds with $s = j$ and $t = i+1$.

Case 4 $i-1$ incides in v and $i+1$ incides in u.

Proceed as in Case 3 by interchanging u with v. ■

Lemma 2.2 *Let G be a graph, $L(G)$ its line graph and $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $L(G)$. If there exists i, $0 \le i \le n-1$ such that $\{[i-1, i+1], [i, i+2]\} \subseteq E(L(G))$, then*

$$\{[i-1, i+2], [i, i+3], [i+1, i+3], [i-2, i], [i-2, i+1]\} \cap E(L(G)) \neq \emptyset\,.$$

Proof: Let $i = [u, v] \in E(G)$ be given. We will consider the following possible cases:

Case 1 $i-1$ incides in u and $i+1$ incides in u.

Let z be the endpoint of $i+1$ different from u. Since $[i, i+2] \in E(L(G))$ we have that $i+2$ incides in u or $i+2$ incides in v. When $i+2$ incides in u we obtain $[i-1, i+2] \in E(L(G))$. When $i+2$ incides in v, the other endpoint of $i+2$ is z. If $i+3$ incides in v we have $[i, i+3] \in E(L(G))$ and if $i+3$ incides in z we obtain $[i+1, i+3] \in E(L(G))$.

Case 2 $i-1$ incides in v and $i+1$ incides in v.

This case follows as Case 1 by interchanging u with v.

Case 3 $i-1$ incides in u and $i+1$ incides in v.

Call z the endpoint of $i-1$ different from u; since $[i-1,i+1] \in E(L(G))$ we have that z is the endpoint of $i+1$ different from v. When $i-2$ incides in u we obtain $[i-2,i] \in E(L(G))$, and when $i-2$ incides in z we have $[i-2,i+1] \in E(L(G))$.

The case 4 when $i-1$ incides in v and $i+1$ incides in u follows as Case 3 by interchanging u with v. ■

We say that a graph H satisfies the property $\mathcal{C}^*$ if and only if for each cycle $\mathcal{C} = (0,1,\ldots,n-1,0)$ the two following properties hold:

(1) If $[i,j] \in E(H) - E(\mathcal{C})$ with $j \notin \{i-2,i+2\}$, then at least one of the following conditions holds:

(1.a) $\{[s-1,s+1],[s,t]\} \subseteq E(H)$ with; ($s=i$ and $t \in \{j-1,j+1\}$) or ($s=j$ and $t \in \{i-1,i+1\}$).

(1.b) $\{[i-1,i+1],[j-1,j+1]\} \subseteq E(H)$.

(1.c) $H[\{s-1,s,t,t+1\}] \cong K_4$ with $s \in \{i,i+1\}$, $t \in \{j-1,j\}$.

(2) If there exists i, $0 \le i \le n-1$ such that $\{[i-1,i+1],[i,i+2]\} \subseteq E(H)$, then

$$\{[i-1,i+2],[i,i+3],[i+1,i+3],[i-2,i],[i-2,i+1]\} \cap E(H) \neq \emptyset\,.$$

Lemma 2.3 *Let H be a graph satisfying the property $\mathcal{C}^*$, and D an orientation of H such that each directed triangle has two symmetrical arcs. If every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0,1,\ldots,n-1,0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has a chord (i,j) such that at least one of the two following properties holds:*

(i) *$j \notin \{i-2,i+2\}$ or*

(ii) *if $j \in \{i-2,i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r,s) with $(r,s) \neq (j,i)$,*

then every directed cycle of D, $\overrightarrow{\mathcal{C}}$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has two symmetrical arcs, ($k \geq 3$).

Proof: Let H and D as in the hypothesis of Lemma 2.3, and take a directed cycle $\overrightarrow{\mathcal{C}}$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$.

We proceed by induction on $\ell(\overrightarrow{\mathcal{C}})$.

If $\ell(\overrightarrow{\mathcal{C}}) = 2$ clearly the two arcs of $\overrightarrow{\mathcal{C}}$ are symmetrical. When $\ell(\overrightarrow{\mathcal{C}}) = 3$ it follows from the hypothesis that $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. Assume that each directed cycle $\overrightarrow{\mathcal{C}}'$ of D with $\ell(\overrightarrow{\mathcal{C}}') < n$ and $\ell(\overrightarrow{\mathcal{C}}') \not\equiv 0 \pmod k$ has two symmetrical arcs. Let $\overrightarrow{\mathcal{C}} = (0,1,\ldots,n-1,0)$ be a directed cycle of D with $\ell(\overrightarrow{\mathcal{C}}) = n$, $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$), (In that follows we will denote by $\mathcal{C}$ the underlying cycle of $\overrightarrow{\mathcal{C}}$), and let (i,j) be a chord of $\overrightarrow{\mathcal{C}}$ such that at least one of the two properties (i), (ii) in the hypothesis of Lemma 2.3 holds. We consider the two possible cases:

Case 1 $j \notin \{i-2,i+2\}$.

Considering $\mathcal{C}$ and $[i, j] \in E(H)$ we have from (1) in property $\mathcal{C}^*$ that at least one of the three properties (1.a), (1.b) or (1.c) holds.

Case 1.a Assume property (1.a) holds; four possibilities will be analyzed.

Case 1.a.1 $\{[i-1, i+1], [i, j-1]\} \subseteq E(H)$. (Considering $s = i$ and $t = j-1$).

Consider the following cycles in H; $\mathcal{C}_1 = [i, j] \cup [j, \mathcal{C}, i]$, $\mathcal{C}_2 = [j, i] \cup [i, \mathcal{C}, j]$, $\mathcal{C}_3 = [i, j-1] \cup [j-1, \mathcal{C}, i]$, $\mathcal{C}_4 = [j-1, i] \cup [i, \mathcal{C}, j-1]$ (see fig. 1). (Notice that $\ell(\mathcal{C}_i) < n$ for each $i \in \{1, 2, 3, 4\}$).

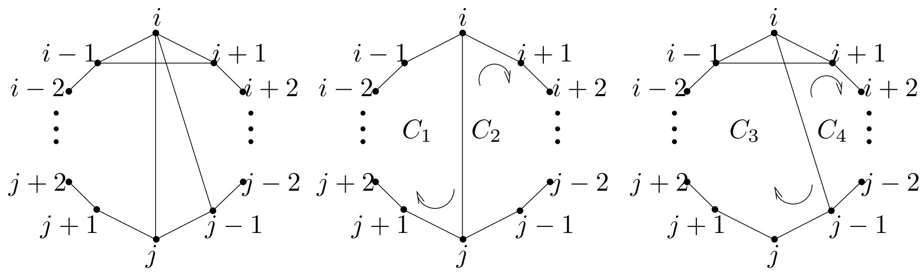


Figure 1:

Case 1.a.1.1 $\ell(\mathcal{C}_1) \not\equiv 0 \pmod{k}$ and $\ell(\mathcal{C}_2) \not\equiv 0 \pmod{k}$.

When $(i, j) \in \mathrm{Sym}(D)$ we have $\overrightarrow{\mathcal{C}}_1 = (i, j) \cup (j, \overrightarrow{\mathcal{C}}, i)$ and $\overrightarrow{\mathcal{C}}_2 = (j, i) \cup (i, \overrightarrow{\mathcal{C}}, j)$ are directed cycles of D; the inductive hypothesis implies that each of them has two symmetrical arcs. Hence $A(i, \overrightarrow{\mathcal{C}}, j) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and $A(j, \overrightarrow{\mathcal{C}}, i) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and then $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $(i, j) \in A(\mathrm{Asym}(D))$ (resp. $(j, i) \in A(\mathrm{Asym}(D))$ we obtain $\overrightarrow{\mathcal{C}_1}$ (resp. $\overrightarrow{\mathcal{C}_2}$)) is a directed cycle of D and the inductive hypothesis implies it has two symmetrical arcs which clearly are symmetrical arcs of $\overrightarrow{\mathcal{C}}$.

Case 1.a.1.2 $\ell(\mathcal{C}_3) \not\equiv 0 \pmod{k}$ and $\ell(\mathcal{C}_4) \not\equiv 0 \pmod{k}$.

Proceed as in Case 1.a.1.1 by changing $[i, j]$ by $[i, j-1]$, $\mathcal{C}_1$ by $\mathcal{C}_3$ and $\mathcal{C}_2$ by $\mathcal{C}_4$.

Case 1.a.1.3 $\ell(\mathcal{C}_1) \equiv 0 \pmod{k}$.

In this case $\ell(\mathcal{C}_3) \equiv 1 \pmod{k}$ and since $k \geq 3$ then $\ell(\mathcal{C}_3) \not\equiv 0 \pmod{k}$. Thus it follows from Case 1.a.1.2 that we can assume $\ell(\mathcal{C}_4) \equiv 0 \pmod{k}$. Since $\ell(\mathcal{C}_1) \equiv \ell(\mathcal{C}_4) \equiv 0 \pmod{k}$ we have $\ell(\mathcal{C}) \equiv -1 \pmod{k}$, and the hypothesis $k \geq 3$ implies $\ell(\mathcal{C}) \not\equiv 1 (\mathrm{mod}\, k)$.

Now consider the edge $[i-1, i+1]$. When $(i-1, i+1) \in A(\mathrm{Sym}(D))$, the directed cycle $(i-1, i+1) \cup (i+1, \overrightarrow{\mathcal{C}}, i-1)$ has two symmetrical arcs. Then $A\left((i+1, \mathcal{C}, i-1)\right) \cap A(\mathrm{Sym}(D)) \neq \emptyset$; also we have that the directed triangle $(i-1, i, i+1, i-1)$ has two symmetrical arcs which implies $A(i-1, i, i+1) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and so $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $(i-1, i+1) \in A(\mathrm{Asym}(D))$ the directed cycle $(i-1, i+1) \cup (i+1, \overrightarrow{\mathcal{C}}, i-1)$ has two symmetrical arcs which also are symmetrical arcs of $\overrightarrow{\mathcal{C}}$. When $(i+1, i-1) \in A(\mathrm{Asym}(D))$, so the two symmetrical arcs of the directed triangle $(i-1, i, i+1, i-1)$ are symmetrical arcs of $\overrightarrow{\mathcal{C}}$.

Case 1.a.1.4 $\ell(\mathcal{C}_2) \equiv 0 \pmod{k}$.

In this case we have $\ell(\mathcal{C}_4) \not\equiv 0 \pmod{k}$ and in view of Case 1.a.1.2 we can assume $\ell(\mathcal{C}_3) \equiv 0 \pmod{k}$. Hence $\ell(\mathcal{C}_1) \equiv -1 \pmod{k}$ and $\ell(\mathcal{C}_1) \not\equiv 0 \pmod{k}$ as $k \geq 3$.

If $(i,j) \in A(\mathrm{Asym}(D))$ (resp. $(j-1,i) \in A(\mathrm{Asym}(D))$), then the directed cycle $\overrightarrow{\mathcal{C}}_1 = (i,j) \cup (j, \overrightarrow{\mathcal{C}}, i)$ (resp. $\overrightarrow{\mathcal{C}}_4 = (j-1,i) \cup (i, \overrightarrow{\mathcal{C}}, j-1)$) has two symmetrical arcs which also are symmetrical arcs of $\overrightarrow{\mathcal{C}}$.

So we can assume $\{(j,i),(i,j-1)\} \subseteq A(D)$ and we obtain the directed triangle $(j,i,j-1,j)$ which has two symmetrical arcs. When $\{(j,i),(i,j-1)\} \subseteq A(\mathrm{Sym}(D))$, then $\overrightarrow{\mathcal{C}_1}$ and $\overrightarrow{\mathcal{C}_4}$ have two simmetrical arcs. Hence $A(j, \overrightarrow{\mathcal{C}}, i) \cap A(\mathrm{Sym}(D)) \neq \emptyset$, $A(i, \overrightarrow{\mathcal{C}}, j-1) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $\{(j,i),(j-1,j)\} \subseteq A(\mathrm{Sym}(D))$ we have $A(j, \overrightarrow{\mathcal{C}}, i) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ (as $\overrightarrow{\mathcal{C}_1}$ has two symmetrical arcs), and then $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $\{(i,j-1),(j-1,j)\} \subseteq A(\mathrm{Sym}(D))$ we have $A(i, \overrightarrow{\mathcal{C}}, j-1) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ (as $\overrightarrow{\mathcal{C}_4}$ has two symmetrical arcs) and then $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs.

Case 1.a.2 $\{[i-1,i+1],[i,j+1]\} \subseteq E(H)$ (here we have $s = i$ and $t = j+1$).

Proceed as in Case 1.a.1 changing $j-1$ by $j+1$.

Case 1.a.3 $\{[j-1,j+1],[j,i-1]\} \subseteq E(H)$, (here we have $s = j$ and $t = i-1$).

Proceed as in Case 1.a.1 by interchanging i with j.

Case 1.a.4 $\{[j-1,j+1],[j,i+1]\} \subseteq E(H)$, (here $s = j$ and $t = i+1$).

Proceed as in Case 1.a.2 by interchanging i with j.

Case 1.b Asssume that property (1.b) holds (i.e. $\{[i-1,i+1],[j-1,j+1]\} \subseteq E(H)$ (see fig. 2)).

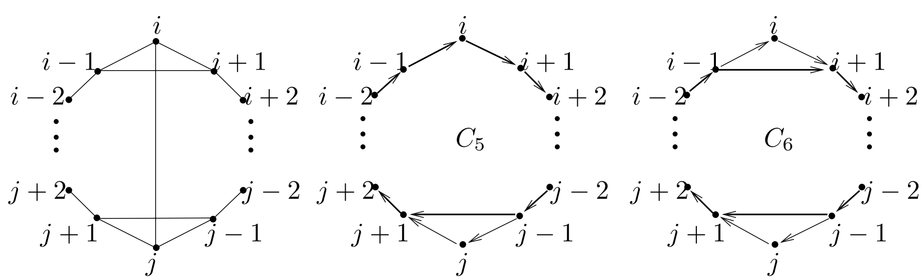


Figure 2:

When $\{(i+1,i-1),(j+1,j-1)\} \subseteq A(D)$ we have the directed triangles $(i-1,i,i+1,i-1)$ and $(j-1,j,j+1,j-1)$ each of them has two symmetrical arcs. Hence $A(i-1,i,i+1) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and $A(j-1,j,j+1) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs.

Assume that $(j-1,j+1) \in A(\mathrm{Asym}(D))$ (When $(i-1,i+1) \in A(\mathrm{Asym}(D))$ proceed exactly as follows only by interchanging j with i).

Let $\overrightarrow{\mathcal{C}_5} = (j-1, j+1) \cup (j+1, \overrightarrow{\mathcal{C}}, j-1)$; when $\ell(\overrightarrow{\mathcal{C}_5}) \not\equiv 0 \pmod{k}$ (see fig. 2) we have that $\overrightarrow{\mathcal{C}_5}$ has two symmetrical arcs which also are symmetrical arcs of $\overrightarrow{\mathcal{C}}$. Suppose that $\ell(\overrightarrow{\mathcal{C}_5}) \equiv 0 \pmod{k}$. If $(i+1, i-1) \in A(\mathrm{Asym}(D))$, then the directed triangle $(i-1, i, i+1, i-1)$ has two symmetrical arcs which also are symmetrical arcs of $\overrightarrow{\mathcal{C}}$. So we can assume $(i-1, i+1) \in A(D)$, and consider the directed cycle $\overrightarrow{\mathcal{C}_6} = (j-1, j+1) \cup (j+1, \overrightarrow{\mathcal{C}}, i-1) \cup (i-1, i+1) \cup (i+1, \overrightarrow{\mathcal{C}}, j-1)$ (see fig. 2); we have $\ell(\mathcal{C}_6) \not\equiv 0 (\bmod\, k)$, hence has two symmetrical arcs; when $(i-1, i+1) \in A(\mathrm{Asym}(D))$ the symmetrical arcs of $\overrightarrow{\mathcal{C}_6}$ are symmetrical arcs of $\overrightarrow{\mathcal{C}}$; when $(i-1, i+1) \in A(\mathrm{Sym}(D))$ we have $A((i-1, i, i+1)) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ and $A((j+1, \overrightarrow{\mathcal{C}}, i-1) \cup (i+1, \overrightarrow{\mathcal{C}}, j-1)) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ which implies $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs.

Case 1.c Assume that property (1.c) holds: Here we have four possibilities; we will analyze the first one and give intructions of how analyze the other three.

Case 1.c.1 $H[\{i-1, i, j-1, j\}] \cong K_4$ (Here we are considering $s = i$, and $t = j-1$).

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ be the cycles of H defined in Case 1.a.1 (see fig. 3).

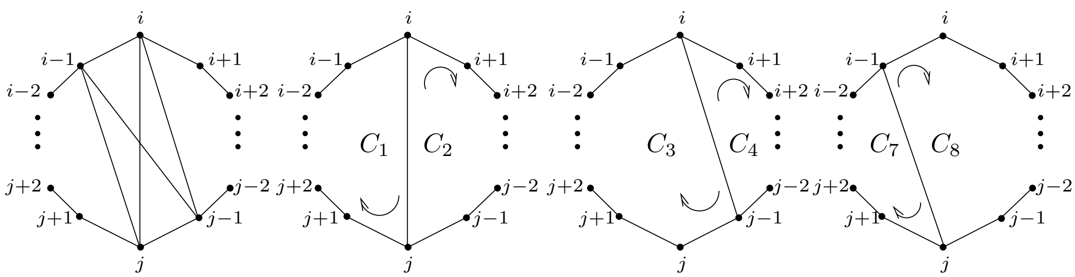


Figure 3:

Case 1.c.1.1 $\ell(\mathcal{C}_1) \not\equiv 0 \pmod{k}$ and $\ell(\mathcal{C}_2) \not\equiv 0 \pmod{k}$.

Proceed as in Case 1.a.1.1.

Case 1.c.1.2 $\ell(\mathcal{C}_3) \not\equiv 0 \pmod{k}$ and $\ell(\mathcal{C}_4) \not\equiv 0 \pmod{k}$.

Proceed as in Case 1.a.1.2.

Case 1.c.1.3 $\ell(\mathcal{C}_1) \equiv 0 \pmod{k}$.

In this case $\ell(\mathcal{C}_3) \not\equiv 0 \pmod{k}$ and from Case 1.c.1.2 we can assume $\ell(\mathcal{C}_4) \equiv 0 \pmod{k}$.

Consider the cycles of H, $\mathcal{C}_7 = [i-1, j] \cup [j, \mathcal{C}, i-1]$ and $\mathcal{C}_8 = [j, i-1] \cup [i-1, \mathcal{C}, j]$ (since fig. 3). Since $\ell(\mathcal{C}_1) \equiv 0 \pmod{k}$ we have $\ell(\mathcal{C}_7) \equiv -1 \pmod{k}$ and $k \geq 3$ implies $\ell(\mathcal{C}_7) \not\equiv 0 \pmod{k}$. The fact $\ell(\mathcal{C}_4) \equiv 0 \pmod{k}$ implies $\ell(\mathcal{C}_8) \equiv 2 \pmod{k}$ and $k \geq 3$ implies $\ell(\mathcal{C}_8) \not\equiv 0 \pmod{k}$.

When $(i-1, j) \in A(\mathrm{Sym}(D))$ we have $\overrightarrow{\mathcal{C}_7} = (i-1, j) \cup (j, \overrightarrow{\mathcal{C}}, i-1)$ and $\overrightarrow{\mathcal{C}}_8 = (j, i-1) \cup (i-1, \overrightarrow{\mathcal{C}}, j)$ each has two symmetrical arcs and then $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $(i-1, j) \in A(\mathrm{Asym}(D))$ (resp. $(j, i-1) \in A(\mathrm{Asym}(D))$), $\overrightarrow{\mathcal{C}_7}$ (resp. $\overrightarrow{\mathcal{C}_8}$) has two symmetrical arcs which are symmetrical arcs of $\overrightarrow{\mathcal{C}}$.

Case 1.c.1.4. $\ell(\mathcal{C}_2) \equiv 0 (\mathrm{mod}\, k)$. Proceed exactly as in Case 1.a.1.4.

When $H[\{i-1, i, j, j+1\}] \cong K_4$ $(s = i, t = j)$ define cycle $\mathcal{C}'_1, \mathcal{C}'_2, \mathcal{C}'_3$ and $\mathcal{C}'_4$ as in Case 1.c.1 by changing $j-1$ by $j+1$, define $\mathcal{C}'_7$ and $\mathcal{C}'_8$ as in Case i.c.1 by changing j by $j+1$, and proceed as in Case 1.c.1 (changing $j-1$ by $j+1$).

When $H[\{i, i+1, j-1, j\}] \cong K_4$ $(s = i+1, t = j-1)$ consider the cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ as in Case 1.c.1; define cycles $\mathcal{C}'_7$ and $\mathcal{C}'_8$ as in Case 1.c.1 by changing $i-1$ by $i+1$, j by $j-1$ and proceed as in Case 1.c.1.

When $H[\{i, i+1, j, j+1\}] \cong K_4$ $(s = i+1, t = j)$ consider the cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ defined in Case 1.a.1 and cycles $\mathcal{C}'_3$, $\mathcal{C}'_4$ obtained by changing $j-1$ by $j+1$ in $\mathcal{C}_3$ and $\mathcal{C}_4$ respectively; finally define cycles $\mathcal{C}'_7$ and $\mathcal{C}'_8$ as $\mathcal{C}_7$ and $\mathcal{C}_8$ respectively by changing $i-1$ by $i+1$ and proceed as in the proof of Case 1.c.1.

Case 2 $j \in \{i-2, i+2\}$.

In this case the hypothesis on Lemma 2.3 imply that there exists another chord of $\overrightarrow{\mathcal{C}}$, (r, s) such that $(r, s) \neq (j, i)$. In view of Case 1 we can assume that there exists a, b $a \neq b$; $\{a, b\} \subseteq \{0, 1, \ldots, n-1\}$ such that $\{[a-1, a+1], [b-1, b+1]\} \subseteq E(H) - E(\mathcal{C})$. When $b-1 \neq a$ proceed as in Case 1.b.

When $b-1 = a$ we have that $\{[a-1, a+1], [a, a+2]\} \subseteq E(H)$ and it follows from property $\mathcal{C}^*$ (2) that:

$\{[a-1, a+2], [a, a+3], [a+1, a+3], [a-2, a], [a-2, a+1]\} \cap E(H) \neq \emptyset$.

We will analyze the five possible cases:

Case 2.1 $[a-1, a+2] \in E(H)$.

Let $\mathcal{C}_1 = [a-1, a+1] \cup [a+1, \mathcal{C}, a-1]$ and $\mathcal{C}_2 = [a-1, a+2] \cup [a+2, \mathcal{C}, a-1]$ (see fig 4).

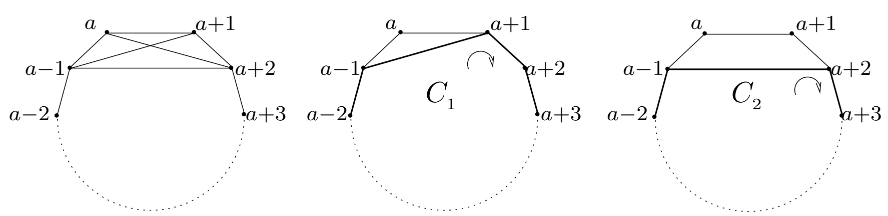


Figure 4:

Case 2.1.1 $\ell(\mathcal{C}_1) \not\equiv 0 \ (\mathrm{mod}\, k)$.

When $(a-1, a+1) \in A(\mathrm{Sym}(D))$ the directed triangle $(a+1, a-1, a, a+1)$ has a symmetrical arc in $(a-1, a, a+1)$ and the directed cycle $\overrightarrow{\mathcal{C}_1} = (a-1, a+1) \cup (a+1, \overrightarrow{\mathcal{C}}, a-1)$ has a symmetrical arc in $(a+1, \overrightarrow{\mathcal{C}}, a-1)$ hence $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $(a-1, a+1) \in A(\mathrm{Asym}(D))$ the two symmetrical arcs of $\overrightarrow{\mathcal{C}_1}$ are in $(a+1, \overrightarrow{\mathcal{C}}, a-1)$ and so $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. When $(a+1, a-1) \in A(\mathrm{Asym}(D))$ the two symmetrical arcs of the directed triangle $(a+1, a-1, a, a+1)$ belong to $\overrightarrow{\mathcal{C}}$.

Case 2.1.2 $\ell(\mathcal{C}_1) \equiv 0 \ (\mathrm{mod}\, k)$.

In this case $\ell(\mathcal{C}_2) \not\equiv 0 \pmod{k}$. First notice that we can assume $(a-1, a+1) \in A(D)$ (when $(a+1, a-1) \in A(\mathrm{Asym}(D))$, the two symmetrical arcs of the directed triangle $(a+1, a-1, a, a+1)$ belong to $\overrightarrow{\mathcal{C}}$).

When $(a-1, a+2) \in A(\mathrm{Sym}(D))$ we have $A((a+2, \overrightarrow{\mathcal{C}}, a-1)) \cap A(\mathrm{Sym}(D)) \neq \emptyset$. Now we consider the directed triangle $(a+2, a-1, a+1, a+2)$. When $(a+1, a+2) \in A(\mathrm{Sym}(D))$ it follows that $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs and when $(a-1, a+1) \in A(\mathrm{Sym}(D))$ we have $A((a-1, a, a+1)) \cap A(\mathrm{Sym}(D)) \neq \emptyset$ which implies $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs.

When $(a-1, a+2) \in A(\mathrm{Asym}(D))$ the directed cycle $\overrightarrow{\mathcal{C}_2} = (a-1, a+2) \cup (a+2, \overrightarrow{\mathcal{C}}, a-1)$ has two symmetrical arcs which belong to $\overrightarrow{\mathcal{C}}$.

When $(a+2, a-1) \in A(\mathrm{Asym}(D))$ the two symmetrical arcs of the directed triangle $(a+2, a-1, a+1, a+2)$ are $(a+1, a-1)$ and $(a+1, a+2)$; now the directed triangle $(a+1, a-1, a, a+1)$ has a symmetrical arc in $(a-1, a, a+1)$; this arc and $(a+1, a+2)$ are two symmetrical arcs which belong to $\overrightarrow{\mathcal{C}}$.

Case 2.2 $[a, a+3] \in E(H)$.

Proceed as in Case 2.1 by changing $a-1$ by a, a by $a+1$, $a+1$ by $a+2$ and $a+2$ by $a+3$.

Case 2.3 $[a+1, a+3] \in E(H)$.

Proceed as in Case 1.b by taking $a = i$ and $a+1 = j-1$.

Case 2.4 $[a-2, a] \in E(H)$.

Proceed as in Case 1.b by taking $i-1 = a-2$ and $j-1 = a$.

Case 2.5 $[a-2, a+1] \in E(H)$.

Proceed as in Case 2.1 changing $a-1$ by $a-2$, a by $a-1$, $a+1$ by a and $a+2$ by $a+1$. ■

Lemma 2.4 *Let H be a graph satisfying the property $\mathcal{C}^*$, and D be an orientation of H such that each directed triangle is symmetrical. If each directed cycle of D whose length is $\not\equiv 0 \pmod{k}$ has two chords, then each directed cycle of D whose length is $\not\equiv 0 \pmod{k}$ has two symmetrical arcs, $(k \geq 3)$.*

Proof: Proceed as in the proof of Lemma 2.3; only notice that when the two chords of a directed cycle $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ (with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod{k}$) are of the form (i, j) and (j, i) with $j \in \{i-2, i+2\}$ then the directed triangle $(i-2, i-1, i, i-2)$, (or $(i, i+1, i+2, i)$) is symmetrical and then $\overrightarrow{\mathcal{C}}$ has two symmetrical arcs. ■

Lemma 2.5 *Let G be a graph, $L(G)$ its line graph and D be an orientation of $L(G)$ such that each directed triangle is symmetrical. If each directed cycle of D whose length is $\not\equiv 0 \pmod{k}$ has two chords, then each directed cycle of D whose length is $\not\equiv 0 \pmod{k}$ has two symmetrical arcs, $(k \geq 3)$.*

Proof: It follows from Lemmas 2.1 and 2.2 that $L(G)$ satisfies property $\mathcal{C}^*$, and then apply Lemma 2.4. ■

Theorem 2.6 [5] *Let D be a digraph such that* $\mathrm{Asym}(D)$ *is strongly connected. If every directed cycle of length* $\not\equiv 0 \pmod k$ *has at least two symmetrical arcs then D has a k-kernel,* $(k \geq 2)$.

Theorem 2.7 *Let G be a graph, $L(G)$ its line graph and D be an orientation of $L(G)$ such that* $\mathrm{Asym}(D)$ *is strongly connected and each directed triangle has two symmetrical arcs. If every directed cycle of D,* $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ *with* $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ *has a chord (i, j) such that at least one of the following properties holds.*

(i) $j \notin \{i-2, i+2\}$, *or*

(ii) *if* $j \in \{i-2, i+2\}$, *then there exists another chord of* $\overrightarrow{\mathcal{C}}$, (r, s)*with* $(r, s) \neq (j, i)$,

then D has a k-kernel, $(k \geq 3)$.

Proof: It follows from Lemmas 2.1 and 2.2 that $L(G)$ satisfy property $\mathcal{C}^*$; then apply Lemma 2.3 and Theorem 2.6. ■

Theorem 2.8 *Let G be a graph, $L(G)$ its line graph and D be an orientation of $L(G)$ such that* $\mathrm{Asym}(D)$ *is strongly connected and each directed triangle is symmetrical. If every directed cycle of D whose length is* $\not\equiv 0 \pmod k$ *has two chords, then D has a k-kernel* $(k \geq 3)$.

Proof: It follows from Lemma 2.4 and Theorem 2.6 (as $L(G)$ satisfies the property $\mathcal{C}^*$). ■

Clearly Theorem 2.8 is a particular case in which the feasible extention of Meyniel's Conjecture ennounced in the introduction holds.

Remark 2.9 The hypothesis that the chord (i, j) of $\overrightarrow{\mathcal{C}}$ satisfies at least one of the two properties (i) or (ii) in Theorem 2.7 cannot be omitted. It suffices to consider the graph G_k, $(k \geq 4)$ defined as follows

$$V(G_k) = \{0, 1, \ldots, 2k-1\}, E(G_k) = \{a_i = [i, i+1] \mid i \in \{0, 2, 4, \ldots, 2k-2\}\}$$
$$\cup \{a_i = [i, i+2] \mid i \in \{1, 3, 5, \ldots, 2k-3\}\} \cup \{a_{2k-1} = [2k-1, 1]\}.$$

Denoted by D_k the orientation of $L(G_k)$ where $V(D_k) = E(G_k)$ and

$$A(D_k) = \{(a_i, a_{i+1}) \mid i \in \{0, 1, 2, \ldots, 2k-2\}\} \cup \{(a_i, a_{i+2}) \mid i \in \{1, 3, 5, \ldots, 2k-3\}\}$$
$$\cup \{(a_{2k-1}, a_0), (a_{2k-1}, a_1)\}.$$

Note that D_k has no k-kernel for $k \geq 4$ and D_9 has no 3-kernel.

(See fig. 5, 6, 7)

— Clearly D_k $(k \geq 4)$ has a hamiltonian asymmetrical directed cycle which implies $\mathrm{Asym}(D_k)$ is strongly connected.

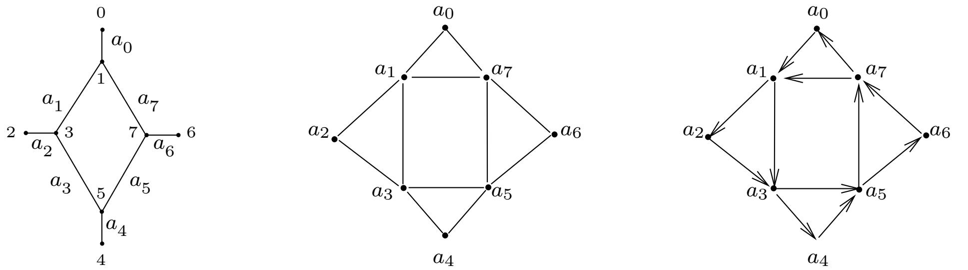


Figure 5:

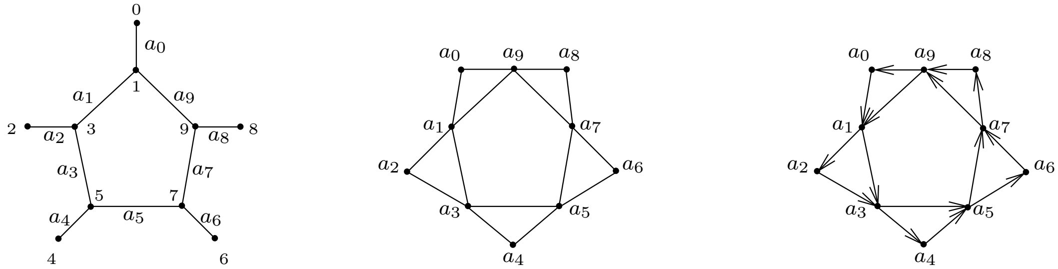


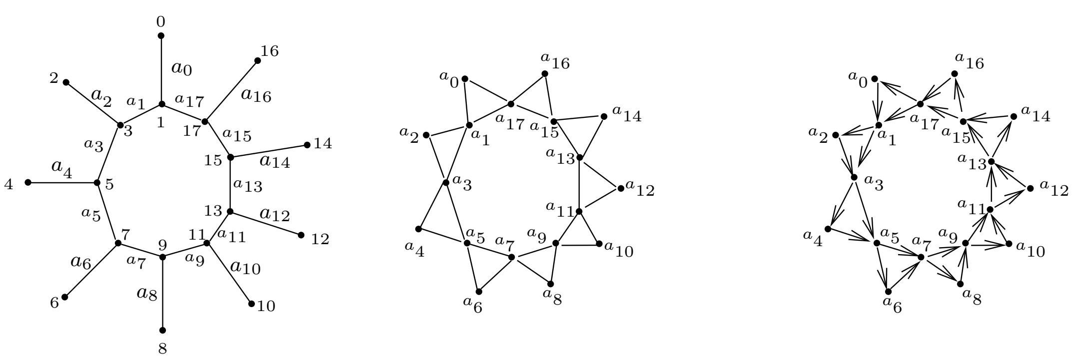
Figure 6:

— D_k has no directed triangles.

— For each $i \in \{0, 2, 4, \dots, 2k-2\}$ the directed cycle $\gamma_i = (a_i, a_{i+1}, a_{i+3}, a_{i+5}, \dots, a_{i+2k-1}, a_i)$ has length $k+1 \not\equiv 0 \pmod{k}$; it has only the chord (a_{i+2k-1}, a_{i+1}) and does not satisfy the hypothesis on Theorem 2.7.

— D_k has no k-kernel: Suppose that D_k has a k-kernel $\mathfrak{N}$; it follows from the definition of k-kernel that $\mathfrak{N} = \{a_i\}$ for some $i \in \{0, 1, \dots, 2k-1\}$; clearly $d_{D_k}(a_{i+1}, a_i) = k$ and so there is no $y \in \mathfrak{N}$ suh that $d(a_{i+1}, y) \leq k-1$. We conclude that $\mathfrak{N}$ is not a k-kernel, a contradiction.

Remark 2.10 The hypothesis that D is an orientation of a line graph cannot be dropped in Theorem 2.7. In the digraph D_k every directed cycle of



G_9 $L(G_9)$ D_9

Figure 7:

length $\not\equiv 0 \pmod k$ has two chords, however D_k has no k-kernel, $(k \geq 3)$.

For D_3 (resp. D_4) see fig. 8 (resp. fig. 9).

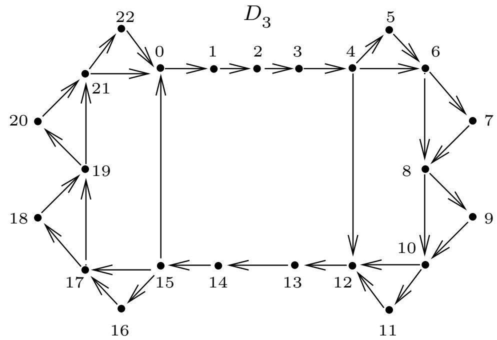


Figure 8:

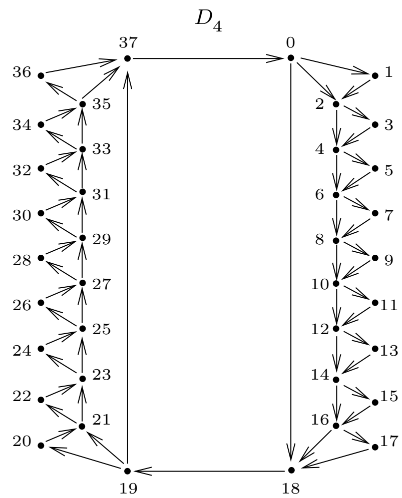


Figure 9:

For $k \geq 5$, first we define H_k as follows:

$$\begin{aligned} V(H_k) &= \{1,2,3,\ldots,3k\}. \\ A(H_k) &= \{(i,i+1) \mid i \in \{1,2,\ldots,3k-1\} \cup \\ & \quad \cup \left\{(3k,1),(3k,k+1),(k+\lfloor\frac{k}{2}\rfloor, 3k+1-\lceil\frac{k}{2}\rceil)\right\} \end{aligned}$$

Clearly each directed cycle of H_k has length$\equiv 0 \pmod k$.

Denote by $B = \{1,\ldots,k+1\} \cup \{k+\lfloor\frac{k}{2}\rfloor+1,\ldots,3k+1-\lceil\frac{k}{2}\rceil$ and now define D_k as follows:

$$\begin{aligned} V(D_k) &= V(H_k) \cup \{s_i, w_i \mid i \in B\} \\ A(D_k) &= A(H_k) \cup \{(i-1,s_i),(s_i,w_i),(w_i,i),(s_i,i) \mid i \in B\} \end{aligned}$$

— Each directed cycle of D_k whose length is $\not\equiv 0 \pmod{k}$ has at least two chords: Notice that if γ is a directed cycle of H_k with $\ell(\gamma) \not\equiv 0 \pmod{k}$, then $s_i \in V(\gamma)$ for some $i \in B$, so γ has at least two chords: $(i-1, i)$ and one of the two following; $(3k, k+1)$ or $(k+\lfloor\frac{k}{2}\rfloor, 3k+1-\lceil\frac{k}{2}\rceil)$.

— Clearly $\mathrm{Asym}(D_k)$ is strongly connected.

— D_k has no k-kernel: Suppose by contradiction that D_k $(k \geq 5)$ has a k-kernel N. First notice that $d_{D_k}(i-1, x) = d_{D_k}(s_i, x)$ for each $x \in V(D_k) - \{s_i, i-1, w_i\}$ and $d_{D_k}(s_i, x) = d_{D_k}(w_i, x)$ for each $x \in V(D_k) - \{s_i, w_i\}$.

For each $j \in B \cup \{3k, k+\lfloor\frac{k}{2}\rfloor\}$ we have $\{j, s_j, w_j\} \cap N = \emptyset$; suppose that there exists $y \in \{j, s_j, w_j\} \cap N$ for some $j \in B \cup \{3k, k+\lfloor\frac{k}{2}\rfloor\}$. Now define z as follows: $z = s_{j+1}$ when $y = j$; $z = w_j$; when $y = s_j$ and $z = s_j$ when $y = w_j$. It follows from the previous assertion that $d_{D_k}(z, x) = d_{D_k}(y, x)$ for each $x \notin \{z, y, w_j\}$. Since $y \in N$ and $k \geq 5$ it follows that $z \notin N$ and $w_j \notin N$, thus there exists $x \in N$ such that $d_{D_k}(z, x) \leq k-1$ $(x \neq w_j)$, and the $d_{D_k}(y, x) = d(z, x) \leq k-1$ with $\{y, x\} \subseteq N$, a contradiction.

It follows from the previous assertion that

$$N \subseteq V(\gamma') - \{3k, k+1, k+\lfloor\frac{k}{2}\rfloor, 3k+1-\lceil\frac{k}{2}\rceil\}$$

where

$$\begin{aligned}\gamma' \;=\; & (3k, k+1) \cup (k+1, k+2, \ldots, k+\lfloor\frac{k}{2}\rfloor) \cup (k+\lfloor\frac{k}{2}\rfloor, 3k+1-\lceil\frac{k}{2}\rceil) \\ & \cup (3k+1-\lceil\frac{k}{2}\rceil, 3k+2-\lceil\frac{k}{2}\rceil, \ldots, 3k),\end{aligned}$$

moreover $|N| = 1$ as $\ell(\gamma') = k$, say $N = \{\ell\}$; clearly $s_1 \notin N$ and $d_{D_k}(s_1, \ell) \geq k$; a contradiction.

The proof that D_3 (resp. D_4) has no 3-kernel (resp. 4-kernel) is completely analogous.

For D_5 (resp. D_6) see fig. 10 (resp. fig. 11).

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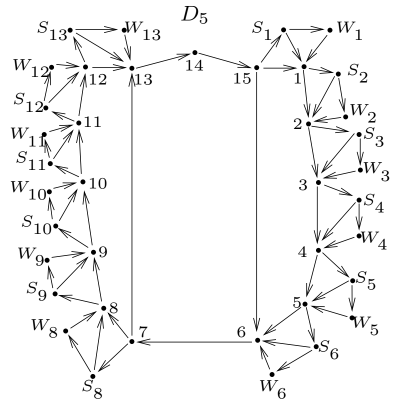

Figure 10:

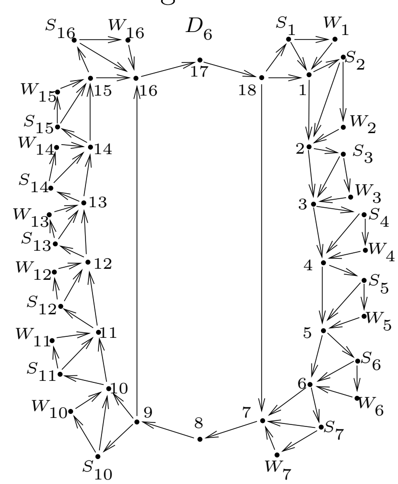

Figure 11:

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