Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 32, 1595 - 1600

Differential Equations on the Half-Line with Discontinuity Conditions

A. Neamaty

Department of Mathematics
University of Mazandaran, Babolsar, Iran
namaty@umz.ac.ir

A. Dabbaghian

Islamic Azad University
Neka Branch, Neka, Iran
a.dabbaghian@umz.ac.ir

Abstract

Boundary value problems for second-order differential equations on the half-line in which the eigenfunctions have a discontinuity in an interior point are investigated. Using of the asymptotic estimates provided in [1] for a special fundamental system of solutions of equation, we study the behavior asymptotic solution and eigenvalues.

Mathematics Subject Classification: 34B24 34B05

Keywords: Differential equations, Asymptotic solution, jump conditions, eigenvalues

1 Introduction

We consider the boundary value problem L for the equation:

$$ly := -y'' + q(x)y = \lambda y, \tag{1}$$

on the the half-line with the boundary condition

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

and with the jump conditions

$$y(d + 0) = ay(d - 0) \qquad , \qquad y'(d + 0) = a^{-1}y'(d - 0). \tag{3}$$

Here $q(x)$ is a real function, h and a are real numbers, $a \neq 1$, $a > 0$ and $d > 0$. Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$, and let for definiteness $\tau := Im\rho \geq 0$. Denote by Π the λ-plane with the cut $\lambda \geq 0$,and $\Pi_1 = \overline{\Pi} - \{0\}$. Then, under the map $\rho \to \rho^2 = \lambda$, Π_1 corresponds to the domain $\Omega = \{\rho : Im\rho \geq 0, \rho \neq 0\}$. Put $\Omega_\delta = \{\rho : Im\rho \geq 0, |\rho| \geq \delta\}$.
Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics,geophysics and other branches of natural properties. For example, it appears in geophysical models for oscillations of the Earth[3,4]. Boundary value problem on the half-line without discontinuities has been studied in [1]. Some aspects for discontinuous boundary value problems in various formulations have been considered in [5-7]. In this paper we will study the solutions and eigenvalues of the boundary value problem L with discontinuities inside the interval on the half-line. In order to study the solutions and eigenvalues in this paper we use the special fundamental system of solutions (FSS) for equation(1). This FSS gives us an opportunity to obtain the asymptotic behavior of the solutions and find characteristic function and eigenvalues. In section 2,using the jump conditions and FSS we get asymptotic from of the solution. In section 3, we study properties of the spectral characteristics of the boundary value problem L.

2 Asymptotic form of the solution

The first, we consider the following theorem that constructed in [1].
Theorem 1. Equation (1) has a unique solution $y = e(x, \rho)$, $\rho \in \Omega$, $x \geq 0$, satisfying the integral equation

$$e(x, \rho) = exp(i\rho x) - \frac{1}{2i\rho}\int_x^\infty (exp(i\rho(x - t)) - exp(i\rho(t - x)))q(t)e(t, \rho)dt. \quad (4)$$

The function $e(x, \rho)$ has the following properties:
1) For $x \to \infty$, $\nu = 0, 1$, and each fixed $\delta > 0$,

$$e^\nu(x, \rho) = (i\rho)^\nu exp(i\rho x)(1 + o(1)) \quad (5)$$

uniformly in Ω_δ.For $Im\rho > 0$, $e(x, \rho) \in L_2(0, \infty)$.Moreover, $e(x, \rho)$ is the unique solution of (1) having this property.
2) For $|\rho| \to \infty$,$\rho \in \Omega$, $\nu = 0, 1$,

$$e^\nu(x, \rho) = (i\rho)^\nu exp(i\rho x)(1 + \frac{\omega(x)}{i\rho} + O(\frac{1}{\rho})), \qquad \omega(x) := -\frac{1}{2}\int_x^\infty q(t)dt, \quad (6)$$

uniformly for $x \geq 0$.
3) For each fixed $x \geq 0$, and $\nu = 0, 1$, the functions $e^\nu(x, \rho)$ are analytic for

$Im\rho > 0$, and are continuous for $\rho \in \Omega$.
4) For real $\rho \neq 0$, the functions $e(x,\rho)$ and $e(x,-\rho)$ form a fundamental system of solutions for (1) , and

$$< e(x,\rho), e(x,-\rho) >= -2i\rho, \tag{7}$$

where $< y, z >:= yz' - zy'$ is the Wronskian.

Let $\psi(x,\lambda)$ be solution (1) under the initial conditions $\psi(1,\lambda) = 1$, $\psi'(1,\lambda) = 0$, and under jump conditions (3). Since the functions $e(x,\rho)$ and $e(x,-\rho)$ form a FSS of (1), then for $d < x$, we have

$$\psi(x,\lambda) = A_1 e(x,\rho) + A_2 e(x,-\rho),$$

that using of Cramer's rule leads to the equation

$$\begin{aligned}\psi(x,\lambda) &= \cos\rho(x-1) + (\omega(x)-\omega(1))\frac{\sin\rho(x-1)}{\rho} \\ &+\frac{1}{2\rho}\int_x^\infty \sin\rho(2t-x-1)q(t)dt \\ &-\frac{1}{2\rho}\int_1^\infty \sin\rho(2t-x-1)q(t)dt + O(\frac{1}{\rho^2}).\end{aligned} \tag{8}$$

In addition, differentiating (8) we calculate

$$\begin{aligned}\psi'(x,\lambda) &= -\rho\sin\rho(x-1) + (\omega(x)-\omega(1))\cos\rho(x-1) \\ &-\frac{1}{2}\int_x^\infty \cos\rho(2t-x-1)q(t)dt \\ &+\frac{1}{2}\int_1^\infty \cos\rho(2t-x-1)q(t)dt + O(\frac{1}{\rho}).\end{aligned} \tag{9}$$

In order to find the solution for $x < d$, we use (3) and Cramer's rule to determine the connection coefficients B_1,B_2 with

$$\psi(x,\lambda) = B_1 e(x,\lambda) + B_2 e(x,-\lambda). \tag{10}$$

Consequently

$$B_1 = \frac{e^{-i\rho d}}{2}[a^{-1}\cos\rho(d-1) + ia\sin\rho(d-1) + O(\frac{1}{\rho})],$$

and

$$B_2(\rho) = B_1(-\rho). \tag{11}$$

By substituting (11) in (10) we derive

$$\psi(x,\lambda) = a^{-1}\cos\rho(d-1)\cos\rho(x-d) - a\sin\rho(d-1)\sin\rho(x-d) + O(\frac{1}{\rho})(12)$$

In addition,differentiating(12) we calculate

$$\psi'(x,\lambda) = -\rho a^{-1}\cos\rho(d-1)\sin\rho(x-d) - a\sin\rho(d-1)\cos\rho(x-d) + O(1)(13)$$

3 Distribution of the eigenvalues

Let $y(x)$ and $z(x)$ be continuously differentiable functions on [0,d] and [d,T]. Denote $< y, z >:= yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the jump conditions (1),then

$$< y, z > |_{x=d+0} \quad = \quad < y, z > |_{x=d-0}, \tag{14}$$

i.e. the function $< y, z >$ is continuous on [0,T]. Let the function $\varphi(x,\lambda)$ be the solution of Eq.(1) that satisfies the initial conditions $\varphi(0,\lambda) = 1$ and $\varphi'(0,\lambda) = h$ and the jump conditions (1). Denote

$$\Delta(\lambda) \quad = \quad < \psi(x,\lambda), \varphi(x,\lambda) > .$$

By virtue (14) and Liouville's formula for Wronskian, $\Delta(\lambda)$ dose not depend on x. The function $\Delta(\lambda)$ is called the characteristic function of L.
Clearly,

$$\Delta(\lambda) \quad = \quad -U(\psi). \tag{15}$$

Definition 2. The values of the parameter λ, for which equation (1) has non-trivial solutions satisfying the conditions $U(y) = 0$, $y(\infty) = 0$ (i.e.$\lim_{x\to\infty} y(x) = 0$), are called eigenvalues of L, and the corresponding solutions are called eigenfunctions.

Lemma 3. Eigenvalues of the problem L are simple, that is, $\Delta_1(\lambda_n) \neq 0$ where, $\Delta_1(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.
proof. Since

$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), \quad -\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n\varphi(x,\lambda_n),$$

we get

$$\frac{d}{dx} < \psi(x,\lambda), \varphi(x,\lambda_n) >= (\lambda - \lambda_n)\psi(x,\lambda)\varphi(x,\lambda_n),$$

hence

$$(\lambda - \lambda_n)\int_0^\infty \psi(x,\lambda)\varphi(x,\lambda_n)dx =< \psi(x,\lambda), \varphi(x,\lambda_n) > (|_0^d + |_d^\infty).$$

If jump conditions (3) and $\alpha_n = \int_0^\infty \varphi^2(x,\lambda_n)dx$ are considered then

$$\int_0^\infty \psi(x,\lambda_n)\varphi(x,\lambda_n)dx = -\Delta_1(\lambda_n),$$

as $\lambda \to \lambda_n$ is obtained. For all $x \in [0,\infty)$, we get from existing of constants β_n(see,[1]) which satisfy the equality $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$ that

$$\alpha_n\beta_n = -\Delta_1(\lambda_n).$$

It is obvious that $\Delta_1(\lambda_n) \neq 0$. So Lemma is proved.
Note that in [1] some properties of eigenvalues have been studied. The main goal of this section is to study asymptotic behavior the zeros of $\Delta(\lambda)$. It follows from (12),(13) and (15) that

$$\Delta(\lambda) =$$

$$\rho(a^+ \sin\rho - a^- \sin\rho(2d-1)) - h(a^+ \cos\rho - a^- \cos\rho(2d-1)) + O(1) \quad (16)$$

Using (16), by the well-known methods (see,for example ,[2]) on can obtain the following property of the eigenvalues $\lambda_n = \rho_n^2$ of the boundary value problem L:

Lemma 4. Let $\lambda_n^o = (\rho_n^o)^2$ be zeros of the function

$$\Delta^o(\lambda) = \rho(a^+ \sin\rho - a^- \sin\rho(2d-1)). \quad (17)$$

Then

$$\rho_n = \rho_n^o + \varepsilon_n, \quad \varepsilon_{n\to 0}. \quad (18)$$

Theorem 5. Eigenvalues of problem L have the following asymptotic behavior

$$\rho_n = \rho_n^o + \frac{\theta_n}{\rho_n^o} + \frac{\kappa_n}{\rho_n^o} \quad (19)$$

where $\kappa_n \in l_2$, and

$$\theta_n = h(a^+ \cos\rho_n^o - a^- \cos\rho_n^o(2d-1))(2\Delta_1^o(\lambda_n^0))^{-1}.$$

proof.We get, substituting (18) into the relation $\Delta(\lambda_n) = 0$ as $\varepsilon_n \to \infty$.

References

[1] G. Freiling and V. Yurko,*Inverse Sturm-Liouville Problems and Thier Applications,* Huntington, New York, 2001.

[2] R. Bellmann and K. Cooke, Differential-difference Equations, *J. Math. Anal. Appl.,***317**(2006), 163-176.

[3] R. S. Anderssen, The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of tortional eigenfrequencies of the Earth,*Geophys. J. R. Astr. Soc.,* **50**(1997), 303-309.

[4] F. R. Lapwood, T. Usami,*Free Oscillations of the Earth, Cambridge Univ. Press*, Cambridge, 1981.

[5] O. H. Hald, Discontinuous inverse eigenvalue problems,*Comm. Pure Appl. Math.,* **37**(1984), 539-577.

[6] A. McNabb, R. S. Andersson, E. R. Lapwood, Asymptotic behavior of the eigenvalues of a Sturm-Liouville system with discontinuous coefficients,*J. Math. Anal. Appl.,* **54**(1976), 741-751.

[7] R. Carlson, An inverse spectral problem for Sturm-Liouville with discontinuous coefficients,*Proc. Amer. Math. Soc.,* **2**(1994), 475-484.

Received: July 25, 2007