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Subschemes of Multi-Projective Spaces and the Generators of Their Multi-Homogeneous Ideal

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Abstract. Let $Y \subset \Pi := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ be a closed subscheme and F a vector bundle on Π. Here we give Castelnuovo-Mumford's style results on the multi-degrees of generators of the multi-graded module $\oplus_{(a_1,\dots,a_k)} H^0(\Pi, \mathcal{I}_Y \otimes F(a_1,\dots,a_k))$.

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1. Introduction

Fix integers $k \ge 2$, $n_i > 0$, $1 \le i \le k$, and an infinite field K. Let $Z \subset \Pi := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ be a closed subscheme. Choose homogeneous coordinates $x_{i,j}$, $1 \le i \le k$, $0 \le j \le n_i$ of each factor of Π. Let e_i, $1 \le i \le k$ denote the basic vector $(0,\dots,1,\dots,0)$ of $\mathbb{N}^k$. Set $\deg(x_{i,j}) = e_i$ for all i,j. With these degrees the polynomial ring $R := K[x_{1,0},\dots,x_{k,n_k}]$ is an $\mathbb{N}^k$-graded ring which will be called the multi-graded ring of Π ([2]). For any $\underline{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ the K-vector space $H^0(\mathcal{O}_\Pi(\underline{a}))$ is the set of all $f \in R$ with multi-degree $\underline{a}$. For any $\underline{a} = (a_1,\dots,a_k) \in \mathbb{Z}^k$ and $\underline{b} = (b_1,\dots,b_k) \in \mathbb{Z}^k$ we will write $\underline{a} \le \underline{b}$ (resp. $\underline{a} \ge \underline{b}$) if and only if $a_i \le b_i$ (resp. $a_i \ge b_i$) for all i.

To extend some of the results of [2] we will first prove some general results (see Theorem 1 and Proposition 1).

Theorem 1. *Let Y be an integral projective variety, $Z \subset Y$ a zero-dimensional subscheme, M, R spanned line bundles on Y and F a coherent sheaf on Y which is locally free outside Z_{red}. Let $V \subseteq H^0(Y,M)$ (resp. $W \subseteq H^0(Y,R)$) be a linear subspace spanning M (resp. R). Let $\mu_W : W \otimes H^0(Y, F \otimes M) \to H^0(Y, F \otimes M \otimes R)$ and $\mu : V \otimes H^0(Y, \mathcal{I}_Z \otimes F \otimes R) \oplus W \otimes H^0(Y, \mathcal{I}_Z \otimes F \otimes M) \to H^0(Y, \mathcal{I}_Z \otimes A \otimes M \otimes R)$ denote the multiplication maps. Assume*

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$h^1(Y, \mathcal{I}_Z \otimes F \otimes R \otimes M^*) = 0$, $h^1(Y, \mathcal{I}_Z \otimes F \otimes M \otimes R^*) = 0$ and that μ_W is surjective. Then μ is surjective.

Remark 1. Notice that in the statement of Theorem 1 we allow the case $M = R$. Hence Theorem 1 and Proposition 1 below cover both [2], Theorem 3, and [2], Theorem 7, and extend them to the case in which Z is not reduced.

Then we will prove the following result on higher dimensional subvarieties of Π.

Theorem 2. *Let $T \subset \Pi$ be a pure c-dimensional subscheme, $0 < c < \sum_{i=1}^{k} n_k$. Let F be a locally free sheaf on Π and $\underline{d}$ such that $h^i(\Pi, F(\underline{u})) = h^i(T, F(\underline{u})|T) = 0$ for all $i > 0$ and all $\underline{u} \ge \underline{d}$. Let $\Pi' \subseteq \Pi$ be the multi-projective space spanned by T. Assume that for every projection α of Π' onto one of its factors, say $\mathbf{P}^s$, $\dim(\alpha(E)) = \min\{s, c\}$ for all irreducible components E of T_{red}. Fix any $\underline{a} = (a_1, \dots a_k) \ge \underline{d}$ such that $h^t(\Pi, \mathcal{I}_T \otimes F(\underline{m})) = 0$ for all pairs $(t, \underline{m} = (m_1, \dots, m_k))$ such that $1 \le t \le c+1$ and $a_i - t + 1 \le m_i \le a_i$ for all $1 \le i \le k$. Then the R-module $\oplus_{\underline{b} \ge \underline{d}} H^0(\Pi, \mathcal{I}_Y \otimes F(\underline{b}))$ is generated by its components with multi-degree $\underline{b} = (b_1, \dots, b_k)$ such that $b_i \le a_i + c + 1$ for all $1 \le i \le c+1$.*

2. Proofs and Proposition 1.

Proposition 1. *Fix an integer $t \ge 0$. Let Y be an integral projective variety, $Z \subset Y$ a zero-dimensional subscheme, a spanned line bundle M on Y and F a coherent sheaf on Y which is locally free at each point of Z_{red} and $h^1(Y, F \otimes M^{\otimes z}) = 0$ for all $z \ge t+1$. Set $V := H^0(Y, M)$. Fix a general $D \in |M|$ and let W be the image of V in $H^0(D, M|D)$. For all integers x let $\mu_x : V \otimes H^0(Y, \mathcal{I}_Z \otimes A \otimes M^{\otimes x}) \to H^0(Y, \mathcal{I}_Z \otimes A \otimes M^{\otimes (x+1)})$ denote the multiplication map. Assume $h^1(Y, \mathcal{I}_Z \otimes A \otimes M^{\otimes x}) = 0$ for $x = t, t+1$ and that for all integers $x \ge t$ the multiplication maps $\alpha_x : V \otimes H^0(Y, A \otimes M^{\otimes x}) \to H^0(Y, A \otimes M^{\otimes (x+1)})$ and $\beta_x : W \otimes H^0(D, A \otimes M^{\otimes x}|D) \to H^0(D, A \otimes M^{\otimes (x+1)}|D)$ is surjective. Then $h^1(Y \mathcal{I}_Z \otimes A \otimes M^{\otimes x}) = 0$ for all $x \ge t+2$ and μ_x is surjective for all $x \ge t+1$.*

Proof. By induction on t it is sufficient to do the case $x = t+2$ of the vanishing statement and the surjectivity of μ_{t+1}. Since M is spanned, D is general and Z is zero-dimensional, $D \cap Z_{red} = \emptyset$. Hence for all integers y there is an exact sequence

$$(1) \qquad 0 \to \mathcal{I}_Z \otimes F \otimes M^{\otimes (y-1)} \to \mathcal{I}_Z \otimes F \otimes M^{\otimes y} \to (F \otimes M^{\otimes y})|D \to 0$$

From (1) and induction on x we get the vanishing statement. Now we will prove that μ_{t+1} is surjective. Let $f \in H^0(Y, M)$ be an equation of D. Fix $u \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M^{\otimes (t+2)})$ and consider $u|D$. Since β_{t+1} is surjective, there are $w_i \in W$ and $u_i \in H^0(D, A \otimes M^{\otimes (t+1)}|D)$ such that $u|D = \sum w_i u_i$. Since W is the image of V and $h^1(Y, \mathcal{I}_Z \otimes A \otimes M^{\otimes (t+1)}) = 0$, there are $v_i \in V$ and $a_i \in H^0(Y, \mathcal{I}_Z \otimes A \otimes M^{\otimes t+2})$ such that $v_i|D = w_i$ and $a_i|D = u_i$. Hence

$(u - \sum v_i a_i)|D \equiv 0$. Since $h^1(Y, \mathcal{I}_Z \otimes F \otimes M^{\otimes t}) = 0$, (1) for $y = t+1$ gives the existence of $v \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M^{\otimes(t+1)}))$ such that $u - \sum v_i a_i = fv$. □

Proof of Theorem 1. Fix a general $f \in V$ and let $D := \{f = 0\}$ denote the associated effective Cartier divisor of Y. Since Z_{red} is finite, V spans M and f is general in V, $D \cap Z = \emptyset$. Hence for all integers α, β we have an exact sequence

(2)
$$0 \to \mathcal{I}_Z \otimes F \otimes M^{\alpha-1} \otimes R^{\beta} \to \mathcal{I}_Z \otimes F \otimes M^{\alpha} \otimes R^{\beta} \to (F \otimes M^{\alpha} \otimes R^{\beta})|D \to 0$$

Let $W_D \subseteq H^0(D, R|D)$ denote the image of W. Fix $u \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M \otimes R)$. Since $h^1(Y, F \otimes M) = 0$, the restriction map $H^0(Y, F \otimes M \otimes R) \to H^0(D, (F|D) \otimes (M|D) \otimes (R|D))$ is surjective. Thus the surjectivity of μ_W gives the surjectivity of the multiplication map $\mu_{W,D} : W_D \otimes H^0(D, (F|D) \otimes (M|D)) \to H^0(D, (F|D) \otimes (M|D) \otimes (R|D))$. Thus there are finitely many $A_i \in W_D$ and $B_i \in H^0(D, (F|D) \otimes (M|D)$ such that $u|D = \sum_i A_i B_i$. Take $A'_i \in W$ such that $A'_i|D = A_i$. Since $h^1(Y, \mathcal{I}_Z \otimes F \otimes M \otimes R^*) = 0$, there is $B'_i \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M)$ such that $B'_i|D = B_i$. Hence $(u - \sum_i A'_i B'_i)|D \equiv 0$. Since $h^1(Y, \mathcal{I}_Z \otimes F \otimes R \otimes M^*) = 0$, there is $G \in H^0(Y, \mathcal{I}_Z \otimes F \otimes R)$ such that $u - \sum_i A'_i B'_i = fG$ (use (2) with $\alpha = 0$ and $\beta = 1$). Hence $u \in \mathrm{Im}(\mu)$. □

Proof of Theorem 2. Without losing generality we may assume $\Pi' = \Pi$. We will use induction on c, the case $c = 0$ being true by Theorem 1. First assume $c = 1$ and take $i \in \{1, \dots, k\}$ such that $\dim(\pi_i(E)) > 0$ for all irreducible components E of T_{red}. Fix a general hyperplane $H \subset \mathbf{P}^{n_i}$ and set $T_1 := T \cap \pi_i^{-1}(H)$. Bertini's theorem ([1], Th. 6.3) gives that T_1 is zero-dimensional. Since $\dim(T) = 1$ and each $\mathcal{O}_{\Pi}(e_i)$ is spanned, if $h^1(T, (F|T)(\underline{b})) = 0$, then $h^1(T, (F|T)(\underline{c})) = 0$ for all $\underline{c} \geq \underline{b}$. If $\underline{b} \in \mathbb{N}^k$, then $h^1(T, \mathcal{O}_Y(\underline{b})) = 0$ if and only if $h^2(\Pi, \mathcal{I}_T(\underline{b})) = 0$. Since $\dim(T_1) = 0$, if $h^1(\Pi, \mathcal{I}_T(\underline{b})) = 0$, then $\underline{b} \in \mathbb{N}^k$ and $h^1(\Pi, \mathcal{I}_T(\underline{c})) = 0$ for all $\underline{c} \geq \underline{b}$. Assume $h^1(\Pi, F(\underline{u})) = 0$ for all $\underline{u} \geq \underline{0} := (0, \dots, 0)$. Since $\dim(T_1) = 0$, if $h^1(\Pi, \mathcal{I}_T \otimes F(\underline{b})) = 0$ and $\underline{b} \in \mathbb{N}^k$, then $h^1(\Pi, \mathcal{I}_T(\underline{c})) = 0$ for all $\underline{c} \geq \underline{b}$. We have an exact sequence

$$(3) \qquad 0 \to \mathcal{I}_T \otimes F(\underline{a} - e_i) \to \mathcal{I}_T \otimes F(\underline{a}) \to \mathcal{I}_{T_1, \Pi_1} \otimes (F|\Pi_1)(\underline{a}) \to 0$$

From (3) we get that the following conditions are equivalent:

(a) $h^1(\Pi, \mathcal{I}_T \otimes F(\underline{a})) = 0$ and the map $i_{2,\underline{a}} : h^2(\Pi, \mathcal{I}_T \otimes F(\underline{a} - e_i)) \to h^2(\Pi, \mathcal{I}_T \otimes F(\underline{a}))$ is injective;

(b) $h^1(\Pi, \mathcal{I}_T \otimes F(\underline{a})) = 0$ and $h^1(\Pi, \mathcal{I}_{T_1} \otimes F(\underline{a})) = 0$.

From now on we assume $h^1(\Pi, F(\underline{u})) = 0$ for all $\underline{u} \geq \underline{0} := (0, \dots, 0)$. If $\underline{a} \geq \underline{0}$ and $h^1(\Pi, \mathcal{I}_{T_1} \otimes F(\underline{a})) = 0$, then $h^1(\Pi, \mathcal{I}_{T_1} \otimes F(\underline{b})) = 0$ for all $\underline{b} \geq \underline{a}$. Hence if $\underline{a} \geq \underline{0}$ and (b) is satisfied then $i_{2,\underline{b}}$ is injective for all $\underline{b} \geq \underline{a}$. Since $(1, \dots, 1)$ is ample, we get that if (a) holds and $\underline{a} \geq \underline{0}$, then $h^2(\Pi, \mathcal{I}_T \otimes F(\underline{b} - e_i)) = 0$ for all $\underline{b} \geq \underline{a}$. Now assume $\underline{a} \geq \underline{0}$ and that (a) is satisfied. Theorem 1 and Proposition 1 give that the R-module $T_1(F, \underline{a}) := \oplus_{\underline{b} \geq \underline{a}} H^0(\Pi_1, \mathcal{I}_{T_1} \otimes F(\underline{b}))$ is generated by its components in degree $\underline{a}$ and $\underline{a}+e_j$, $1 \le j \le k$. Theorem 1 and Proposition 1 give that the R-module $T(F,\underline{a}) := \oplus_{\underline{b}\ge\underline{a}} H^0(\Pi, \mathcal{I}_T \otimes F(\underline{b}))$ is generated by its components in degree $\underline{a}$, $\underline{a}+e_j$, $1 \le j \le k$, and $\underline{a}+e_j+e_m$, $1 \le j \le m \le k$, proving the case $\dim(T) = 1$. The inductive proof of the general case require only notational modifications and hence it is omitted. □

References

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